SPACE AND TIME
DEŚA-KĀLA

KALĀTATTVAKOŚA
VOLUME – II

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NEW DELHI
and
MOTILAL BANARSIDASS PUBLISHERS PVT. LTD.
DELHI

1992[1] 2003[2]

ŚŪNYA/ŚŪNYATĀ

Overview

Śūnya means void, devoid of, empty, zero, and *śūnyatā* denotes the doctrine of emptiness propounded by Buddhism. It is the emptiness of space which is a condition for the appearance of any object, just as the number zero, being no number at all, is the condition for all the numbers. Therefore zero (*śūnya*) and infinite (*ananta*) have been identified.

I. In mathematical usage *śūnya* denotes zero which has been taken as the basis of the decimal system. India's contribution to mathematics and science is the discovery of the zero.

II. In Buddhism, *śūnyatā* does not denote nihilism, but the negation of all affirmative statements about the world. It corresponds to a spiritual experience and to the practice of emptying the mind from all impressions. The doctrine of *śūnyatā* has found its full development in Nāgārjuna.

III. The dominant note of the Vedic religion and thought was that of fullness (→*pūrṇa*), but the idea of the void as a spiritual state is equally present in the Hindu Tantras which describe several stages of *śūnya* in the spiritual development (cf. Śaivāgamas and Kashmir Śaivism).

In the Arts the idea of *śūnya/śūnyatā* is important in the context of the aesthetic experience, and in the creative act. The creation of poetry, of a painting or a musical composition has to come out of a vacant state of the mind of the artist. This corresponds to the visualization of a deity in Tāntric Buddhist practice, as described in the **Hevajra Tantra**:

प्रथमं शून्यताबोधिं द्वितीयं बीजसंग्रहम् ।
तृतीयं बिम्बनिष्पत्तिं चतुर्थन्न्यासमक्षरम् ॥

HT I.3.2

First the realization of the void,
secondly the seed in which all is concentrated,
thirdly the physical manifestation,
and fourthly one should implant the syllable. (Tr. D.L. Snellgrove)

Whether the process is that of representing the deity in an image or of visualizing its form in the inner imagination, the same creative process is involved which starts precisely from the state of the void (cf. also **Sādhanamālā**).

The architect who builds a temple has to deal with and organize empty space. It is not the walls which make a building but the empty spaces created

by the walls. As A.K. Coomaraswamy puts it: "In order to understand the use of terms for 'space' (*kha, ākāśa, antarikṣa, śūnya*, etc.) as verbal symbols of zero (which represents privation of number, and is yet a matrix of number in the sense 0 = x - x), it must be realized that *ākāśa*, etc., represent primarily a concept not of physical space, but of a purely principial space without dimensions, though the matrix of dimension." (A.K. Coomaraswamy, "*Kha* and Other Words Denoting 'Zero'...", in: Selected Papers, Vol. 2, p. 225).

Etymology

The word *śūnya* has been derived from *śūna* which Pāṇini recognizes as the past participle of *śvi-*, 'to go, to grow', generally 'to swell'. *Śuna*, then, should mean 'swollen' or 'grown' but in its occurrences in the **Ṛgveda** it also seems to have the sense of 'lack or deficiency'. It is possible that there were two different words originally but being of similar form they were fused giving a single sense of 'absence or emptiness'. But the word *śūnya* does not occur in the classical Upaniṣads except that the probably later **Maitreyī Upaniṣad** (II.27) seeks to identify the *śūnya* with the *pūrṇa*.

The word *śūnya* means void, empty (*śūnyaṃ tu vaśikaṃ tucchariktake*, AmKo III.1.56). The **Amarakośa** gives four synonyms of *śūnya* meaning void, empty, worthless and absent. This is illustrated by Cāṇakya in the following verse:

> अविद्याजीवनं शून्यं दिक्शून्या चेदबान्धवा ।
> पुत्रहीनं गृहं शून्यं सर्वशून्या दरिद्रता ॥
>
> Cāṇakya Śat 47
>
> Life is useless if one lacks knowledge, the direction is aimless if one does not have any friends and relatives, to one who is without a son his house is desolate, everything is void to the poor. (Tr. H.N. Chakravarty)

In the neuter the term *śūnya* denotes the void of the sky (→*ākāśa, kha, vyoman*).

Editor

I. Śūnya: Mathematical Aspect

In Indian mathematics, *śūnya* denotes zero. The nine numbers (*aṅka*) from 1 to 9, together with the zero, constitute the basis of the decimal place-value system which is universally followed today. It is generally agreed that this system

originated in India. Likewise the ten numeral symbols emanated from India, though they are now known as Arabic numerals after the chief mediators in the transmission of these symbols from India to the western world.

A discussion of the mathematical aspect of *śūnya* involves three questions: the concept and synonyms of *śūnya* within the decimal place-value system, secondly the symbols used for *śūnya* and thirdly its mathematical functions.

Concept of Zero

While the numbers 1 to 9 are common to many other numeral systems, it is the concept of *śūnya* along with its symbol known as *bindu* that makes the Indian numeral system a truly place-value system, without which the great advances made in modern mathematics are unthinkable. Here *śūnya* denotes the notational place as well as the 'void' or the absence of numerical value in a particular notational place. Consequently all numerical quantities, howsoever great they may be, can be represented with just these ten symbols. Writing in the twelfth century, King Someśvara declares in his **Mānasollāsa** thus:

एकाद्या नवपर्यन्ता नवैवाङ्का स्वरूपतः ।
दशोत्तरक्रमेणैव वर्धन्ते बिन्दुवर्धिताः ॥

Mānaso II.97

Basically, there are only nine digits starting from 'one' and reaching up to 'nine'. By the addition of zeros (*bindus*) these are raised successively to (represent the notational places of) tens, (hundreds) and beyond.

In Vedic literature, there are already several sets of names for the numbers, arranged in a progressive geometrical series where each number is ten times larger than the preceding. Thus the Vājasaneyī recension of the **Yajur-Veda** enumerates in the following passage the names of thirteen decimal places, viz. *eka* (10^0), *daśa* (10^1), *śata* (10^2), *sahasra* (10^3), *ayuta* (10^4), *niyuta* (10^5), *prayuta* (10^6), *arbuda* (10^7), *nyarbuda* (10^8), *samudra* (10^9), *madhya* (10^{10}), *anta* (10^{11}) and *parārdha* (10^{12}) :

इमा मे अग्न इष्टका धेनवः सन्तु । एका च दश च, दश च शतं च, शतं च सहस्रं च, सहस्रं चायुतं चायुतं च नियुतं च, नियुतं च प्रयुतं चार्बुदं च न्यर्बुदं च समुद्रश्च मध्यं चान्तश्च परार्धश्चैता मे अग्न इष्टका धेनवः सन्त्वमुत्रामुष्मिँल्लोके ।

YV XVII.2

O Agni, may these bricks(used in sacrifice) turn into (milk-yielding) cows for me; one(*eka*) and a ten, a ten(*daśa*) and a hundred, a hundred (*śata*) and a thousand, a thousand (*sahasra*) and a ten-thousand, a ten

thousand(*ayuta*) and a hundred-thousand, a hundred-thousand(*niyuta*) and a million(*prayuta*), a ten-million (*arbuda*), a hundred-million (*nyarbuda*), a thousand-million (*samudra*), ten-thousand-million (*madhya*), a hundred thousand-million (*anta*), a million-million (*parārdha*). May these bricks become mine own cows in this and in the other world.

Obviously this decuple terminology gave rise to the decimal place-value system, but it is difficult to say how old the place-value system is and when the concept of zero originated. Perhaps the genesis has to be sought in some type of abacus or counting board with columns or spaces for decimal notational places. The word *gaṇitra*, a *hapax legomenon* occurring in the Jyotiṣka story of the **Divyāvadāna** (Darbhanga 1959, p.163) may have denoted such a counting board. Emptiness in any column meant the absence of numerical value in that particular notational place, and this emptiness was indicated by the word *śūnya* or by one of its synonyms.

The synonyms commonly used for *śūnya* in mathematical and astronomical texts are *ākāśa*, *ambara*, *kha*, *gagana* and the like for the *ākāśa* is conceived to be emptiness. Occasionally *ananta* is also used, not in the sense of the infinite but as another synonym for *ākāśa*. Other synonyms for *śūnya* are →*pūrṇa*, →*bindu*, *chidra* and *randhra*. The last three actually describe the symbols used for *śūnya*, viz. a small dot (*bindu*) and a small circle as used today (*chidra*, *randhra*).

The earliest mention of a symbol for zero occurs in the **Chandaḥ-Sūtra** of Piṅgala. In this text, Piṅgala teaches a method for calculating the number of arrangements of long and short syllables in a metre containing a certain number of syllables. In modern mathematical parlance, this is the number of combinations of two things in 'n' places, repetition being allowed, which is equal to 2^n. Piṅgala lays down the procedure in short obscure aphorisms:

द्विरर्धे । रूपे शून्यम् । द्विः शून्ये । तावदर्धे गुणितम् ।

Chandaḥsūtra 8.28-31

(First write down the number of syllables in the given metre and go on halving that number. Each time) when (the number is) halved, (write down in a separate row or column the digit) two.

(When you reach an odd number, subtract one from it.) Whenever one (is subtracted, write down in the separate row a) zero.

(Continue thus until the process stops. Then where you wrote a) zero (multiply by) two.

Where (the number was) halved, multiply (the result of the second process) by itself.

It is beyond the scope of this article to explain Piṅgala's method in detail. What is relevant is the fact that in this computation Piṅgala uses the symbols for zero and two as markers for distinguishing between two kinds of operations. The symbol of two marks the places where there is an even number which is divided by 2 and where squaring has to be done later; the symbol of zero marks the places where there is an odd number and consequently absence of halving, and where multiplication by 2 has to be performed. Thus the two symbols were used here in a meaningful way, but the whole computation can be done without any markers at all or with any two arbitrary symbols. The fact, however, that Piṅgala uses them goes to show the existence of a well-recognised symbol for zero at his time, which is variously placed between 400 to 200 B.C.

Joseph Needham observes that "Place-value could and did exist without any symbol for zero... But the zero symbol as part of the numerical system never existed and could not have come into being without place-value" (Science and Civilisation in China, vol.III, p.10n). We do not know what kind of zero symbol was used at Piṅgala's time, but the invention of the place-value system along with concept and symbol of zero must antedate considerably his mention of the zero symbol. A clear reference, however, to the place-value is made in the Jaina canonical text **Anuyogadvāra** (ca. 100 B. C., but this date is contested) where the number of human beings in the world is stated to occupy twentynine notational places (*sūtra* 142: *egūṇatīsaṃ-ṭhāṇāi*=Sanskrit, *ekonatriṃśat sthānāni*).

Thus there is enough indirect evidence to say that the decimal place-value system with symbols for 1 to 9 and zero developed in India much before the beginning of the Christian era. But in the early centuries of this era epigraphic records still continued to use the old system of numerals without place-value.

In the extant mathematical and astronomical literature, however, place-value is implicit throughout, even if it is not directly employed. Thus in his **Āryabhaṭīya** (A.D. 499). Āryabhaṭa-I uses a complex numerical notation but his method of extracting square-roots and cube-roots is possible only in the decimal place-value system. Again, in the following stanza where he explains his peculiar method of expressing numbers with the letters of the alphabet, he clearly mentions the zero symbol:

खद्विनवाङ्के स्वरा नव वर्गेऽवर्गे ...

Āryabhaṭīya I.2

In the place of two nines of zeros (which are written to denote the notational places), the nine vowels should be written (one vowel in each pair of *varga* and *avarga* places)... (Tr. K.S. Shukla)

Āryabhaṭa's alphabetical system of notation is not of immediate relevance here, but his instruction to write down nine pairs of zeros (*kha*) first, in order

to denote 18 notational places is interesting. This alludes to an old practice as Bhāskara-I explains in his commentary on the **Āryabhaṭīya** (II.2). In fact, even in recent times, children were taught the decimal notational places by means of a row of zeros written in the following manner:

0 0 0 0 0 0 0 0 0 0

The teacher pointed to the first zero on the right and explained that it was the units place, the next zero as the tens place, the next as the hundreds place and so on (Datta and Singh, History of Hindu Mathematics: A Source Book, p. 82).

In other texts there are unambiguous instances where the word *śūnya* or its synonyms were used in the decimal place-value system. A few examples may be given here. About a century prior to the **Āryabhaṭīya**, the **Puliśasiddhānta** is said to have contained the following stanza, now available only as a quotation in Bhaṭṭotpala's commentary on Varāhamihira's **Bṛhatsaṃhitā** (Varanasi, Vol. I, p. 27):

ख-ख्याष्ट-मुनि-रामाश्वि-नेत्राष्ट-शर-रात्रिपाः ।
भानां चतुर्युगेणैते परिवर्त्ताः प्रकीर्तिताः ॥

The sidereal revolutions in four *yugas* are stated to be (the number expressed by) zero (*kha*), zero (*kha*) eight (*aṣṭa*), seven (*muni*), three (*rāma*), two (*aśvin*), two (*netra*), eight (*aṣṭa*), five (*śara*), and one (*rātripa*) (read serially from the left, i.e. 1,582,237,800).

A clear and unambiguous mention of the zero in association with the decimal place-value occurs in Sarvanandi's **Lokavibhāga** (Ch.4, Line 56, p.79):

पञ्चभ्यः खलु शून्येभ्यः ।
परं द्वे सप्त चाम्बरमेकं त्रीणि च रूपं च ॥

After five zeros, there are two (*dve*), seven(*sapta*), zero(*ambara*), one(*eka*), three(*trīṇi*) and one(*rūpa*), (that is to say, 13,107,200,000).

The **Lokavibhāga** mentions the date of its completion as Śaka 380 bhādra kṛṣṇa 15, which corresponds to Monday, 25th August 458 A. D. This, then is the earliest dated occurrence of the zero known to us.

In Varāhamihira's **Pañcasiddhāntikā** (ca A.D. 550), *śūnya* and its synonyms occur often, as in I.17:

मुनि-यम-यम-द्वियुक्ते द्युगणे शून्य-द्वि-पञ्च-यम-भक्ते ।
प्रतिराशि खर्तु-दहनैर्लब्धं वर्षाणि यातानि ॥

Increase the *ahargaṇa* by 2227 and divide (the sum) by 2510; with respect to the (remaining) amount, divide it by 360; the quotient is

the number of lapsed years... (Tr. D. Pingree)

In his **Bṛhat-kṣetra-samāsa** (A.D. 609), Jinabhadra Gaṇi Kṣamāśramaṇa offers conclusive evidence of the use of zero as a distinct numerical symbol. When he has to write large numbers such as 224,400,000,000, he expresses them succinctly as "twentytwo, fortyfour, eight zeros" (I.69) or 3,200,400,000,000 as "thirtytwo, two zeros, four, eight zeros"(I.71).

Siddhasena Gaṇi (6th century), in his commentary on Umāsvāti's **Tattvārthādhigamasūtra** (III.11), makes an interesting use of the inherent decimal property of *śūnya*:

शेषमिदं भवति ३,५३४,४००,०००,००० । अस्य मूलमादीयते ।
शून्याष्टकार्धेन चत्वारि शून्यानि शेषस्य मूलमेकाष्टकाष्टकैर्लब्धमिदं १,८८०,००० ।

... the remainder is this: 3,534,400,000,000. The square-root of this is extracted; half of eight zeros are four zeros; the root of the remaining part is one-eight-eight; hence the resulting root is 1,880,000. (Tr. B. Datta and A.V. Singh)

This example shows that Siddhasena understood that the *śūnya* is not only a symbol for the absence of numerical value in a particular notational place but that it also indicates the tenth multiple inherent in the notational place.

By the second century A.D., even outside the realm of mathematics the idea of the decimal place-value and consequently of the numerical value of any digit depending upon the notational place was so well rooted that this fact was used to elucidate philosophical notions. In his **Abhidharmakośabhāṣya**, Vasubandhu(fifth century A. D.) cites Vasumitra's view in the following passage (p.296) :

अवस्थान्यथिको भगन्तवसुमित्रः। स किलाह। धर्मोऽध्वसु प्रवर्तमानोऽवस्थामवस्थां प्राप्यान्यो निर्दिश्यते, अवस्थान्तरतो न द्रव्यान्तरतः। यथैका वर्तिका एकाङ् के निक्षिप्ता एकमित्युच्यते, शताङ्के शतं, सहस्राङ्के सहस्रमिति।

Abhidh Ko Bh V.26

Bhadanta Vasumitra held the view of the otherness of the state. He is reported to have stated (as follows). A dharma evolving in the (three) times is designated as other whenever it enters a different state; (this change in designation is) due to the otherness of the state and not due to the otherness of the substance; just as a marker or counter is called one when it is placed in the unit's place, a hundred in the hundred's place, and a thousand in the thousand's place.

It may be noted that this view is attributed to Vasumitra and the wording purports to be his. According to D. Seyfort Ruegg who discusses this passage, "Vasumitra is ... probably to be identified as one of the leading figures at the time of Kaniṣka's Great Council." ("Mathematical and linguistics Models in Indian Thought," p.173). Accordingly, Vasumitra must have flourished in the last quarter of the first century and the first quarter of the second century A. D.

This metaphor of the changing value of a numeral digit according to its place is employed in several other philosophical texts. Thus it is said in the **Vyāsabhāṣya** (bet. A.D. 650 to 850) on the **Yogasūtra:**

यथैका रेखा शतस्थाने शतं दशस्थाने दशैकं चैकस्थाने तथा चैकत्वेऽपि स्त्री माता चोच्यते दुहिता च स्वसा चेति ।

YSūBh III.13

Just as the same stroke is termed a hundred in the hundreds place, ten in the tens place and one in the units place, so is one and the same woman referred to as mother, daughter or sister.

Vijñānabhikṣu emphasises the role of the *bindu* in determining the numerical value of a digit in his commentary on the above passage:

यथैकत्वव्यञ्जिका रेखा अङ्कविशेषो यदा बिन्दुद्वयोपरि तिष्ठति तदा शतमिदं नैकमिति व्यवह्रियते, तयोरेकबिन्दुलोपे च दशेदं न शतमिति व्यवह्रियते, अवशिष्टबिन्दुस्थाने चैकत्वव्यञ्जकरेखान्तरदाने सत्येकादशेदं न दशेत्यर्थः।

Yogavārttika on YSūBh III.13

Just as, when the special stroke denoting the digit one stands to the left of two zeros, it is known as a hundred and not as one, when one of the two zeros is removed, it is known as ten and not (any more) as hundred, and when in the place of the remaining zero another stroke denoting the number one is marked, it is known as eleven and not ten...

Śaṅkara also makes use of this metaphor in his commentary on the **Brahma-Sūtra:**

यथैकोऽपि सन् देवदत्तो लोके स्वरूपं सम्बन्धिरूपं चापेक्ष्यानेकशब्दप्रत्यय-भाग्भवति मनुष्यो ब्राह्मण: श्रोत्रियो वदान्यो बालो युवा स्थविर: पिता पुत्र: पौत्रो भ्राता जामातेति तथा चैकापि रेखा स्थानान्यत्वेन निविशमानैक-दश-शत-सहस्रादिशब्दप्रत्ययभेदमनुभवति ।

BrSūBh II.2.17

Devadatta, although being one only, forms the object of many different names and notions according as he is considered in himself or in his relation to others: thus he is thought and spoken of as man, *brāhmaṇa*, learned in the Veda, generous, boy, young man, old man, father, son, grandson, brother, son-in-law, etc., so again, one and the same stroke is, according to the place it is connected with, spoken of and conceived as meaning either one, ten or hundred or thousand etc. (Tr. P.J. Deussen)

Symbols of Zero

As we have stated above, of the various synonyms of *śūnya*, *bindu* and *chidra/randhra* refer to two kinds of symbols used for representing the zero. Of these, *bindu* or dot is the earlier one; later its place was taken by a small circle (*chidra* or *randhra*, lit. "hole"). In the case of *anusvāra* also an analogous change occurred in Kannaḍa-Telugu script. In early inscriptions, the *anusvāra* was represented by a small dot, often hardly visible on the copper plates. Gradually it was replaced by the small circle, which is called *sunna* in Telugu (from Sanskrit *śūnya*) since it was identical with the symbol of *śūnya*.

There are no clear evidences to say when the change from the dot to the circle took place. In the extant records, the earliest instance of the zero symbol occurs in the eighth century Rāgholi plates of Jayavardhana II, where a tiny dot was used in writing the number 30. Clearly drawn circles appear in the next century in the Gwalior plates of Bhojadeva belonging to the years A.D. 870 and 876.

Interestingly enough, zero symbols appear some two hundred years prior to this in South-East Asia. In the inscriptions found at Cambodia and Sumatra, the Śaka year 605 (=A.D. 683) is written in the decimal system and the zero is represented by a dot. In another inscription from Banka Island, a small circle was used for recording the date, viz. Śaka 608 (= A.D. 686). Obviously the Indian settlers carried with them both the symbols for zero from India, as they did carry the Sanskrit language, word numerals, decimal system, the Śaka era, and so on.

It may thus be concluded that both the zero symbols were in use in India in the sixth century and that the circular symbol must have been invented by this century at the latest. The change, however, was not simultaneous everywhere.

The dot seems to have prevailed longer in Gandhāra and Kaśmīra regions. Thus in the mathematical text known as the Bakhshālī Manuscript and in the unique manuscript copy of an anonymous commentary on Śrīdhara's **Pāṭīgaṇita**, both written in an early form of the Śāradā script sometime after the ninth century, the zero was represented by the dot.

Even after the dot was replaced by the circle, the symbol continued to be called *śūnya-bindu* or *bindu*, as the passages cited above from the **Mānasollāsa** and **Yogavārttika** show. Therefore, when *śūnya-bindu* is mentioned occasionally in literary works, it is difficult to decide whether the dot or the circle is meant. In Subandhu's **Vāsavadattā** (ca. A.D. 400), we meet with the earliest occurrence of the expression, and here it may have been the dot, since the poet compares the stars in the sky with zeros on a dark writing medium:

> विश्वं गणयतो विधातुः शशिकठिनीखण्डेन तमोमषीश्यामेऽजिन इव वियति संसारस्यातिशून्यत्वाच्छून्यबिन्दव इव विलिखिताः ... तारा अराजन्त ।
>
> Vāsavadattā (Chowkhamba edition, p.168)
>
> ... the stars shone forth, ... like zero dots (*śūnya-bindu*) because of the nullity of metempsychosis, scattered in the sky as if on the ink-blue skin rug of the Creator who reckoneth the sum total with a bit of the moon for the chalk. (Tr. Louis H. Gray)

On the other hand, in Śrīharṣa's (12th century) description of the King Nala, the *śūnya-bindus* may be small circles, as here the pores on the skin are seen as zeros:

> किमस्य लोम्नां कपटेन कोटिभिर्विधिर्न लेखाभिरजीगणद् गुणान् ।
> न रोमकूपौघमिषाज्जगत्कृता कृताश्च किं दूषणशून्यबिन्दवः ॥
>
> Naiṣadhīyacarita I.21
>
> Did not the creator reckon his merits with crores of lines, the hairs of the body? Did not the maker of the world put the pores of his skin for zeros to indicate the absence of defects? (Tr. K.K. Handiqi)

Indian numerals reached the Middle East through the mediation of the Arabs, though nothing definite is known about the precise period and the path. Among the Arabs, these numerals were known as *Hindasā*. Already in 662, the Syrian scholar Severus Sebokht speaks highly of the Hindus' "valuable methods of calculation; and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs." The Indian numerals known to Sebokht must have contained the zero also, for without it and without the implied decimal place-value system, the nine signs would not have been any improvement over Roman numerals.

Through his **Liber Abaci** (A.D. 1202), Leonardo Fibonacci of Pisa popularised the Indian numerals including the zero in Europe. In this passage from India to Europe, the word *śūnya* underwent many transformations: it became *aṣ-ṣifr* or *ṣifr* in Arabic, *zephizum* in Fibonacci's Latin and finally gave rise to the forms 'zero' and 'cipher'.

Mathematical Functions of Zero

Sanskrit texts on mathematics usually contain a section called *śūnya-gaṇita*, i.e. computations involving zero. While the discussion in the arithmetical texts (*pāṭīgaṇita*) is limited only to the addition, subtraction and multiplication with zero, the treatment in algebra (*bījagaṇita*) covers also such questions as the effect of zero on the positive and negative signs, division with zero, and more particularly the relation between zero (*śūnya*) and infinity (*khahara* = *ananta*).

Brahmagupta, an outstanding mathematician and astronomer of the seventh century, was the first to discuss exhaustively the mathematical functions of zero in his **Brāhmasphuṭa-Siddhānta** (18.30-35). By treating the zero as a separate entity from the positive (*dhana*) and negative (*ṛṇa*) quantities, he implies that *śūnya* is neither positive nor negative but forms the boundary line between the two kinds. It is the sum of two equal but opposite quantities (*dhanarṇayoḥ samaikyaṃ kham*). Brahmagupta states that a number, whether positive or negative, remains unchanged when zero is added to or subtracted from it. In multiplication with zero, the product is zero. A zero divided by zero or by some number becomes zero. Likewise the square and the square-root of zero is zero. But when a number is divided by zero, the quotient is an undefinable quantity, which Brahmagupta designates elliptically as *taccheda*, "that which has that (sci. zero) as the denominator".

Though his formulation "zero divided by zero is zero" will not be accepted today, Brahmagupta did in fact cover almost all mathematical functions of *śūnya*. Mathematicians who came after him took up his concept of *taccheda* and developed it further. Thus Śrīpati (fl. 1039-1056) replaces the word *taccheda* with a more expressive term *khahara*, "that which has zero (*kha*) as the divisor":

विकारमायान्ति धनर्णकानि न शून्यसंयोगवियोगतस्तु ।
शून्याद्विशुद्धं स्वमृणं क्षयं स्वं वधादिना खं खहरं विभक्तम् ॥

Siddhāntaśekhara XIV.6

Numbers, both positive and negative, undergo no change when zero is added to or subtracted from them. But when substracted from zero, a positive number becomes negative and a negative number positive. (A number) when multiplied (by zero becomes) zero; and when divided (by zero becomes) *khahara*.

The celebrated Bhāskarācārya of the twelfth century explains that this *khahara* is an infinite quantity (*ananto rāśiḥ*). What then is an infinite number? Bhāskara resorts to a significant analogy while elucidating the nature of *khahara*:

> अस्मिन् विकारः खहरे राशावपि प्रवेष्टेष्वपि निःसृतेषु ।
> बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत् ॥
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> Bījaganita, verse 20
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> In this numerical quantity called khahara, there will be no change whether much is added to it or subtracted from it, just as no change takes place in the infinite and immutable godhead, though numerous beings are absorbed in it at the time of the destruction of the worlds or numerous beings issue forth from it at the time of the creation of the worlds.

Bhāskara's commentator, Kṛṣṇa Daivajña offers, at the beginning of the seventeenth century, the rationale for the multiplication and division with zero. Explaining how in multiplication with zero the product becomes zero, Kṛṣṇa argues as follows:

> गुण्यस्यापचयवशाद् गुणनफलस्यापचय इति तावत्प्रसिद्धम् । ...अनयैव युक्त्या गुण्यस्य परमापचये गुणनफलस्यापि परमापचयेन भाव्यम् । परमापचये शून्यतैव पर्यवस्यतीति शून्ये गुण्ये गुणनफलं शून्यमेवेति सिद्धम् ।...एवं गुणकापचयवशादपि गुणनफलेऽपचयाद् गुणकस्यापि शून्यत्वे गुणनफलं शून्यमेवेति सिद्धम् ।
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> Navāṅkurā Comm. on Bījaganita, p.18
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> It is quite well known that as the multiplicand is diminished, the product also diminishes by the same degree... By this logic, when the multiplicand is diminished to the utmost degree, the product also diminishes to the utmost degree. But the utmost reduction of a number results in its becoming *śūnya*. Hence, when the multiplicand is *śūnya*, it is obvious that the product will also be *śūnya* ... In the same manner, as the multiplier decreases, the product also decreases. Hence, when the multiplier is *śūnya* the product is also *śūnya*.

The same logic applies when a number is divided by zero:

> यथा यथा भाजकापचयस्तथा तथा लब्धेरुपचयः । तथा सति भाजकाङ्के परमापचिते लब्धेः परमोपचयेन भाव्यम् । लब्धेश्चेदियत्तोच्येत तर्हि परमत्वं

न स्यात् ततोऽप्याधिक्यसम्भवात् । अतो लब्धेरियत्ता भाव एव परमत्वम्। तदेवमुपपन्नं खहरो राशिरनन्त इति ।

Navāṅkurā Comm on Bījagaṇita p.19

As the divisor is diminished, the quotient increases in a like manner. This being so, when the divisor is diminished to the maximum limit, the quotient must also increase to the maximum limit. If it is specified that the quotient is so much, then the maximum limit is not reached, for a quantity greater than the (specified one) is possible. Therefore, the quotient reaches the maximum limit only when it is impossible to specify that it is so much. Thus it is established that the quantity called *khahara* is infinity.

Bhāskara's views are not fully developed, however, in one respect. Discussing the multiplication with zero, he states:

...खगुणः खं खगुणश्चिन्त्यश्च शेषविधौ ॥
शून्ये गुणके जाते खं हारश्चेत् पुनस्तदा राशिः ।
अविकृत एव ज्ञेयः ...

Līlāvatī, 45-46

The product of zero is zero. But if there remains an operation still to be performed, imagine (the product) to be a multiple of zero. For, zero having become the multiplier, should zero become afterwards the divisor as well, then the (original) quantity remains unchanged.

That is to say, $\underline{a}.0 = 0$ but $\frac{\underline{a}.0}{0} = \underline{a}$.
This is based on the wrong assumption that the zero in the nominator and that in the denominator cancel out each other, leaving a unchanged. Modern interpreters like Datta and Singh wish to reconcile this position by treating the zero here as the infinitesimal, i.e. a quantity decreasing indefinitely without actually becoming zero, because while $\frac{\underline{a}.0}{0}$ is not $\underline{a}$, $Lim_{\Sigma \to 0} \frac{\underline{a}.\Sigma}{\Sigma} = \underline{a}$.

However, if Bhāskara's intention here is to discuss the infinitesimal, he does not distinguish it from zero, and in any case the whole discussion occurs in the middle of his treatment of zero.

Therefore, Nārāyaṇa Paṇḍita refutes this view of Bhāskara by saying that a quantity, once it becomes zero through multiplication, shall remain in that state even if it is subsequently divided by zero, just as a yogin remains in the state of bliss when once he attains it:

शून्याभ्यासवशात् खतामुपगतो राशिः पुनः खोद्धृतो
व्यावृत्तिं पुनरेव तन्मयतया न प्राकृतिं गच्छति ।
आत्माभ्यासवशादनन्तममलं चिद्रूपमानन्ददं
प्राप्य ब्रह्मपदं न संसृतिपथं योगी गरीमानिव ॥

Bījagaṇitāvataṃsa, p.6

Just as a great yogin, having attained, through meditation on the Ātman, the state of Brahman – a state that is infinite, untainted, blissful and of the nature of pure intelligence –, will not revert to the path of rebirth because he has been absorbed in that (Brahman), even so a (numerical) quantity, having itself become zero after multiplication by zero, will never revert to the original number, for it has been absorbed in that (zero).

S.R. Sarma