THE CONCEPT OF ŚŪNYA

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Śūnya in Piṅgala's Chandaḥsūtra

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mayarasatajabhanalagasaṃmitaṃ bhramati vāṅmayaṃ jagati yasya/
sa jayati piṅgalanāgaḥ śivaprasādād viśuddhamatiḥ//

— Halāyudha[1]

The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, a symbol, but also a helpful power, is the characteristic of the Hindu race whence it sprang. It is like coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power.

— G.B. Halsted[2]

I

The *Chandaḥsūtra* of Piṅgala[3] constitutes an important landmark in the history of the decimal place value system for it mentions, for the first time, zero (*śūnya*) and its symbol. Even though it describes a large number of classical metres in addition to Vedic metres, this text is traditionally considered a Vedāṅga. It is composed in *sūtra* style and contains some 310 *sūtras*, distributed into eight chapters. The descriptive technique employed by Piṅgala has close affinity with that of Pāṇini.[4] Similarity in notation has also been noticed between the *Chandaḥsūtra* and the *Vedāṅgajyotiṣa*. Both these texts employ the first or the final syllable of the names of the *nakṣatras* as their designation.[5] The *Chandaḥsūtra* also introduced a kind of algebraic notation for the classification of prosodic units as *ma-gaṇa, ya-gaṇa* and so on. Significant for the history of numerical notation in India is also the fact that the *Chandaḥsūtra* employs the so-called word-notation or *bhūtasaṃkhyā* quite frequently.

The German indologist Albrecht Weber was the first to make a critical study of the *Chandaḥsūtra* as early as 1863, under the title "Ü ber die Metrik der Inder" in the eighth volume of his journal *Indische Studien*.[6] Since then the text did not receive critical attention, although there is an urgent need of a fresh discussion on the constitution of the text and its chronology.[7]

II

Śūnya is mentioned in the *Chandaḥsūtra* in the context of what are technically known as *pratyayas*.[8] It is, therefore, necessary to give a brief overview of these *pratyayas*. As is well known, the poetic metres in Sanskrit are constituted by the varying arrangements of two kinds of syllables, viz. long syllables (*guru*) and short syllables (*laghu*). *Pratyayas* deal with problems of combinations of these syllabic lengths in a verse-foot (*pāda*). In the last fifteen *sūtras* of the final eighth *adhyāya*, Piṅgala discusses five *pratyayas*, viz. *Prastāra, Naṣṭa, Uddiṣṭa, Saṃkhyā* and *Eka-dvy-ādi-ga-la-kriyā*. In an admirable study, Van Nooten has shown recently how for the first time in history Piṅgala employed here "the binary number as a means for classifying metrical patterns".[9] I may briefly explain each of these *pratyayas*, and against this background dwell at length on the fourth *pratyaya* called *Saṃkhyā* which is relevant to our discussion of *śūnya*.

PRASTĀRA:[10]

It is the arrangement of all possible combinations of *guru* and *laghu* in a given metre. It must be added that Piṅgala deals here only with *samavṛttas*, i.e. metres where all the four feet of the verse have identical patterns. Therefore, he takes into consideration only the verse foot (*pāda*). For example, if the foot of a particular metre contains three syllables, its *Prastāra* will be as shown in following table. Here *g* denotes *guru* and *l* *laghu*.

It will be seen that in the first variation, all the three syllables are *gurus* and that in the last all are *laghus*. How was this table achieved? An examination will show that in the first column, single *guru* (2^0) alternates with single *laghu*; in the second column two *gurus* (2^1) alternate with two *laghus*; and in the third column four *gurus* (2^2) alternate with four *laghus*. This can be continued for any number of syllables in the verse foot. That Piṅgala, however, stops with three syllables has a special reason. These eight variations of three syllables give rise to the so-called *gaṇas* with which the metres are scanned or classified. As I said, Piṅgala designates these eight variations with a kind of algebraic notation: the first variation with all *gurus* is designated *m* or — for the convenience of clear pronunciation — *ma*, or *ma-gaṇa* and so on. In fact, the *Chandaḥsūtra* commences with the definition of these eight *gaṇas* in this very sequence. And now, towards the end, Piṅgala demonstrates how these eight *gaṇas* are achieved.[11]

1.	*g g g*	*m*	*ma*	*ma-gaṇa*
2.	*l g g*	*y*	*ya*	*ya-gaṇa*
3.	*g l g*	*r*	*ra*	*ra-gaṇa*
4.	*l l g*	*s*	*sa*	*sa-gaṇa*
5.	*g g l*	*t*	*ta*	*ta-gaṇa*
6.	*l g l*	*j*	*ja*	*ja-gaṇa*
7.	*g l l*	*bh*	*bha*	*bha-gaṇa*
8.	*l l l*	*n*	*na*	*na-gaṇa*

NAṢṬA:[12]

The second *pratyaya* called *Naṣṭa* is the process of finding out the pattern of a specified variation. Given the serial number of the variation within the *Prastāra*, its pattern has to be found out without actually constructing the *Prastāra*. This is achieved by the continuous halving of the given serial number. Whenever it is divisible by 2, we write down a *laghu* or '*l*'. When it is not divisible, we write down a *guru* or '*g*', add 1 to the number and continue.

For example, we may find out the form of the variation no. 5 in the *Prastāra* for 3 syllables, as we have constructed above. We go on halving the number 5.

5 is odd, write down *g*, add 1 to 5 and halve: $(5 + 1)/2 = 3$

3 is odd, write down *g*, add 1 to 3 and halve: $(3 + 1)/2 = 2$

2 is even, write down *l*, halve: $2/2 = 1$.

Therefore, variation no. 5 has the form *g g l*.

UDDIṢṬA:[13]

In the next *pratyaya* called *Uddiṣṭa*, the pattern is given and its serial number within the *Prastāra* has to be found out. Thus it is just the reverse of *Naṣṭa* and need not detain us.

EKA-DVY-ĀDI-GA-LA-KRIYĀ:[14]

It is the process (*kriyā*) of classifying the variations into those having one *guru*, those having two *gurus* and so on (*eka-dvy-ādi-ga*). For this purpose a pyramid (*meruprastāra*) is constructed in the following manner with $n + 1$ rows, or in the present case with $3 + 1 = 4$ rows.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
$$

The last row provides the answer to the problem. It shows that in the *Prastāra* of 3 syllables there are:

1 form with 3 *gurus* (and 0 *laghu*)
3 forms with 2 *gurus* (and 1 *laghu*)
3 forms with 1 *guru* (and 2 *laghus*)
1 form with 0 *gurus* (and 3 *laghus*).[15]

III

Now we come to the *pratyaya* called *Saṃkhyā* and the mention of *śūnya* therein. This is the method for computing the total number of arrangements of long and short syllables without actually constructing the *Prastāra*. In modern mathematical parlance, this is the number of combinations of two things in '*n*' places, repetition being allowed. There are several ways of solving the problem. We have just seen in connection with the *Prastāra* that the number of combinations equals to 2^n where *n* indicates the number of syllables in the verse foot. Therefore, all one has to do is multiply 2 as many times as there are syllables.[16] Another method is to add up all the numbers in the bottom row of the number pyramid. In the case of three syllables, this will be:

$$1 + 3 + 3 + 1 = 8.$$

But Piṅgala teaches a much more complex method which, as will be shown presently, is of great importance. The method is enunciated in four pithy *sūtras*:

dvir ardhe / rūpe śūnyam / dviḥ śūnye / tāvad ardhe tad guṇitam /[17]

By completing the sentences, these *sūtras* may be rendered into English as follows :

" [First write down the number of syllables in the given metre and go on halving that number. Each time] when [the number is] halved (*ardhe*), [write down in a separate row or column the digit] 2 (*dviḥ*).

"[When you reach an odd number, subtract 1 from it.] Whenever 1 [is subtracted (*rūpe*), write down in a separate column a] zero (*śūnyam*).

"[Continue thus until the process stops. Then where you wrote a] zero (*śūnye*), [multiply by] 2 (*dviḥ*).

"Where [the number was] halved (*tāvad ardhe*), multiply [the result of the second process] by itself (*tad guṇitam*)".

The import of these *sūtras* will be clear from a concrete example. We take the sacred *Gāyatrī* metre which has six syllables in each *pāda*. Required are the number of variations when each one of the six syllables is either a *guru* or a *laghu*. We know already that the answer will be 2^6. But let us see where Piṅgala's method leads to. We proceed as follows:[18]

		A	B	C
1.	Write the number of syllables	6		
2.	Halve 6	3	2	$(2^2.2)^2$
3.	3 cannot be halved; therefore reduce it by 1.	2	0	$2^2.2$
4.	Halve 2.	1	2	2^2
5.	1 cannot be halved; therefore reduce it by 1.	0	0	1.2
		Stop!		Commence!

In column A, the number of syllables is successively halved and, whenever there is an odd number, it is reduced by 1. In column B, on the other hand, we write the two kinds of markers: '2' when halving is possible and '0' when it is not. When halving comes to an end in column A, the process now continues in column C from below, opposite the last

marker in column B. Taking unity, double it whenever there is a '0' in column B and square it whenever there is a '2'. At the top of column C, we obtain the result, viz. $(2^2.2)^2$ which is, of course, equal to 2^6.

The process laid down by Piṅgala reduces the number of operations. In the case of *Gāyatrī*, if we try to compute 2^6, we have to multiply 2 by 2 five times, i.e. it involves five operations. Piṅgala's method involves only three: (i) squaring 2; (ii) multiplying by 2; and (iii) squaring again.[19]

Important for the present discussion, however, is the fact that Piṅgala uses the symbols for zero and two as markers for distinguishing between two kinds of operations. The symbol of two marks the place where there is an even number which is divided by 2 and where squaring has to be done later; the symbol of zero marks the stages where there is an odd number and consequently absence of halving and where multiplication by 2 has to be performed. Thus two symbols were used here in a meaningful way. The whole computation can of course be done without any markers at all or with any two arbitrary symbols. However, the fact that Piṅgala uses these two markers shows that at Piṅgala's time there existed a well recognised symbol for mathematical *śūnya*. A symbol presupposes a concept. What kind of mathematical concept lay behind this symbol for *śūnya*? From Piṅgala's use, it may appear that *śūnya* meant here the absence of an operation, akin to the grammarian's *lopa*. But is that all, or does *śūnya* here imply a place value system as well ?[20]

It is useful in this connection to consider the view of Joseph Needham who states that "Place value could and did exist without any symbol for zero. ... But zero symbol as part of the numerical system never existed and could not have come into being without place value".[21]

Therefore, Piṅgala's employment of zero symbol presupposes place value. That it can only be decimal place value needs no emphasis. Vedic literature is replete with decuple terminology such as *eka, daśa, śata* and so forth. To sum up this discussion, Piṅgala could have used any other marker. But that he used a marker called *śūnya* establishes without any doubt that a symbol for zero was quite well known in his times. We do not know what kind of zero symbol it was, but a zero symbol presupposes place value. Therefore, the invention of decimal place value system along with the concept and symbol of zero must antedate considerably Piṅgala's mention of the zero symbol. But when did Piṅgala live?

IV

In an entry on the "Mathematical Aspect of Śūnya" contributed to the second volume of *Kalātattvakośa*, I noted that Piṅgala's time "is variously placed between 400 to 200 B.C." and therefore concluded that "there is enough indirect evidence to say that the decimal place value system with symbols for 1 to 9 and zero developed in India much before the beginning of the Christian era".[22]

This time-frame was contested by a scholar who argued, following Weber, that the *Chandaḥsūtra* was a late work and that, moreover, the eighth chapter where the zero was mentioned was not original.[23]

Although I am reluctant to join the bandwagon of those who are obsessed with establishing the Indian priority in every aspect in the history of ideas, I must say that these two objections are not tenable. It is admittedly difficult to postulate an absolute chronology for any of the ancient Sanskrit texts, but many can be located in relation to other texts. We have noted the close similarity in the method of exposition and notation between the *Chandaḥsūtra* on the one hand and the *Aṣṭādhyāyī* and the *Vedāṅgajyotiṣa* on the other. The affinity with these two texts places the *Chandaḥsūtra* about 400 B.C. But I shall not press this point. Weber's objection to such an early period rests on the fact that the *Chandaḥsūtra* contains several classical metres in addition to Vedic metres. I believe that a re-examination is necessary of the nineteenth century assumption that there existed a clear line of demarcation between the Vedic and classical periods.[24]

As regards the second objection, we are on much firmer ground. *Pratyayas* are discused in the last 15 *sūtras*, i.e. from 8.20 to 8.34, of the *Chandaḥsūtra*. The passage preceding this, viz. *sūtras* 2-19, does not occur in several manuscripts. Therefore, Weber dismissed the entire eighth *adhyāya* as not original. But the absence of *sūtras* 2-19 in some manuscripts does not establish that *sūtras* 20-34 are not genuine. Indeed the whole problematique of *pratyayas* has more to do with Vedic poetry than with the classical poetry. Unlike the classical poetry, Vedic poetry endows the metres with magical significance.[25] Playing with the arragements of metres is a favourite pastime of the Veda and can be understood only in the context of the magical significance of such variations. In his *Rig-Veda Repetitions*, Maurice Bloomfield has shown that variation in metre is a stylistic device in the Veda and that recurrence of otherwise identical *pādas* is accompanied by changes in the metre, which are mostly effected

by extensions or abbreviations.[26] Moreover, the various modes of Vedic recitation, such as the *padapāṭha, kramapāṭha, jaṭāpāṭha, ghanapāṭha* deal with arrangements of words in a manner not unrelated to the *pratyayas*. Therefore, it is quite certain that the mathematics of the *pratyayas* developed in the context of Vedic prosody[27] just as the geometry of the *Śulvasūtras* developed in the context of Vedic ritual.

But far more important is the following. The *pratyaya*-section of the eighth chapter is not a loose appendage but is anticipated in the preceding chapters of the *Chandaḥsūtra*. The text commences with the definition of the eight *gaṇas* or triplets, viz. *m, y, r, s, t, j, bh, n,* followed by the definition of *l* and *g*. This sequence of triplets makes sense only in the context of the *Prastāra* of the eighth chapter as I have shown in Section II above. Otherwise, Piṅgala could have enumerated the triplets in any other sequence. Again at the beginning of the fifth chapter, Piṅgala defines the *sama-vṛtta, ardha-sama-vṛtta* and *viṣama-vṛtta* and states that the variations in the second and third categories are obtained by squaring the number of variations in the preceding category (5.1-4). That is to say, since the variations in a *sama-vṛtta* of *n* syllables are 2^n, the number of variations in an *ardha-sama-vṛtta* will be $2^n \times 2^n = 2^{2n}$; and in a *viṣama-vṛtta* they will be $2^{2n} \times 2^{2n} = 2^{4n}$. This too is not unrelated to the *pratyayas*. Clearly then the eighth chapter containing the combinatorics and the first mention of *śūnya* is not a late interpolation but an integral part of Piṅgala'ṣ *Chandaḥsūtra*.[28]

To conclude, Piṅgala's mention of *śūnya* is a significant event in the history of ideas. It shows that the decimal place value system with the numbers 1 to 9 and zero developed in India before the beginning of the Christian era.

. REFERENCES

1. At the commencement of his *Mṛtasañjīvanī* commentary on Piṅgala's *Chandaḥsūtra*.
2. *On the Foundation and Technique of Arithmetic*, Chicago 1912, p. 10.
3. *Piṅgalacchandaḥsūtram*, with the *Mṛtasañjīvanī* commentary by Halāyudha and a further commentary by Jīvānanda Vidyāsāgara Bhaṭṭācārya, Calcutta 1928.
4. B. Van Nooten, "Binary Numbers in Indian Antiquity", *Journal of Indian Philosophy* 21, 31-50, 1993, esp. p. 49, n. 4: "We recognise the use of *adhikāras*,

such as '*chandas*' (2.1), the *anuvṛttis*, the use of *śeṣe* (2.12) to include unnamed contexts, the use of ablative to indicate context-after and the locative for context-before. In brief, enough similarities exist to show that the descriptive techniques used by Piṅgala and the grammarians were similar".

5. David Pingree, *Jyotiḥśāstra: Astral and Mathematical Literature* (J. Gonda, ed, *A History of Indian Literature*, Vol. VI, Fasc. 4), Wiesbaden 1981, p. 10: "It (the *Vedāṅgajyotiṣa*) imitates Piṅgala's Chandaḥsūtra in using the final or first syllables of the names of the nakṣatras as their designations; its period relation is copied in the oldest *Paitāmahasiddhānta*, whose epoch is 11 January 80; and its astronomy reflects that of Mesopotamia in the Achaemenid period. It is likely, therefore, that it was composed not very many years before or after 400 B.C. when the Achaemenids controlled Gandhāra".
6. *Indische Studien: Beiträge für die Kunde des indischen Alterthums*, Band VII, Berlin 1863; reprint: Hildesheim/New York 1973.
7. This is all the more necessary after the publication of the *Chandoviciti*, discovered among the collection of birch bark manuscripts found at Turfan; cf. Schlingloff, Dieter, *Chandoviciti: Texte zur Sanskritmetrik* (Sanskrittexte aus den Turfanfunden, hrsg. Ernst Waldschmidt, Band V), Berlin 1958.
8. On the *pratyayas*, see Ludwig Alsdorf's comprehensive study, "Die Pratyayas. Ein Beitrag zur indischen Mathematik", *Zeitschrift für Indologie und Iranistk*, 9, 97-157, 1933; English tr. by Sreeramula Rajeswara Sarma, "The Pratyayas: Indian Contribution to Combinatorics", *Indian Journal of History of Science*, 26.1, 17-61, 1991. Henceforth references will be made to the English translation.
9. B. Van Nooten, *op. cit.*, 32.
10. *Chandaḥsūtra*, 8.20-23.
11. Piṅgala goes on to add that such triplets can only be eight, neither more nor less. The *sūtra* in question (8.23) reads *vasavas trikāḥ*, i.e. "The triplets (*trikāḥ*) are just eight (*vasavaḥ*) in number". Notice here the use of the word numeral *vasavaḥ* which stands for eight. Piṅgala himself draws our attention to this fact in 1.15: *aṣṭau vasava iti*.
12. *Chandaḥsūtra*, 8.24-25.
13. *Ibid.*, 8.26-27.
14. *Ibid.*, 8.33-34.
15. By tilting the pyramid by about 45°, we get the following figure, which is similar to Pascal's triangle, so named after the French philosopher and mathematician Blaise Pascal, who invented the triangle in 1654 :

1	1	1	1
1	2	3	
1	3		
1			

The four numbers in the hypotenuse of this triangle are, however, the binomial coefficients of $(a + b)^3$ when expanded: $(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$.

16. Cf. Hemacandra, *Chando'nuśāsana*, 7.11 : *varṇasamadvikahatiḥ*.
17. *Chandaḥsūtra* 8.28-31. The next *sūtra*, 8.32: *dvir dvyūnaṃ tadantānām* is also related to the *Saṃkhyā*, but is not immediately relevant here. It teaches that the sum of the *saṃkhyās* from 1 syllable in a *pāda* to *n* syllables is the *Saṃkhyā* for *n* syllables, multiplied by 2 and reduced by 2, or $2^1 + 2^2 + 2^3 \ldots + 2^n = 2.2^n - 2$.
18. Cf. Datta and Singh, *op. cit.*, i, pp. 75-77; Alsdorf, *op. cit.*, pp. 38-39; see also Gupta, R.C., "An Ancient Method of Piṅgala for Finding a^n", *Bona Mathematica*, 1, 77-80, 1990.
19. Cf. Alsdorf, *op. cit.*, p. 38: "Piṅgala teaches a method which, at first sight, appears to be strange and complicated, but in reality, is very ingenious. It is based, as Weber already explained, 'on a very ingenious way of simplification of the exponent' which results in the reduction of the operations needed for calculating the *saṃkhyā*. For instance, in order to calculate the value of 2^{11}, we have to multiply 2 ten times with 2. But we can go about it in another way. The modern mathematician will factorize 2^{11} thus :
$$2^{11} = 2^{10}.2 = (2^5)^2.2 = (2^4.2)^2.2 = [(2^2)^2.2]^2.2.$$
To calculate the 1st form, we need just to perform thrice squaring and twice multiplication, i.e. five operations instead of 10. The Indian method leads exactly to this kind of simplification of higher powers into lower with an exponent not larger than 2. To know when the squaring is done and when the multiplication with 2, the exponent (in our example 11) is halved continuously: if we reach an odd number, it is reduced by 1. Where an even number is halved, we have to square later and this is marked with an index number '2'. Where 1 is subtracted, we have to perform multiplication later on with 2, and this is marked with the index number 'zero'".
20. Piṅgala employs word numerals rather frequently: in fact, he employs 15 different words with numerical significance in about 100 instances. But in all these places, he uses the words singly. Therefore, his employment of word numerals does not clearly indicate place value.
21. Needham, Joseph and Ling, Wang, *Science and Civilisation in China*, Vol. II, Cambridge 1959, p. 10 n.
22. Sarma, S.R., "Śūnya, Mathematical Aspect" in: Bettina Bäumer (ed.), *Kalātattvakośa: A Lexicon of Fundamental Concepts of the Indian Arts*, vol. 2, New Delhi 1992, pp. 400-11, esp. 403.
23. Bronkhorst, Johannes, "A Note on Zero and the Numerical Place-Value System in Ancient India", *Asiatische Studien/Études Asiatiques*, 48.4, 1039-42.

1994. In support of the view that the *Chandaḥsūtra* is a late work, Bronkhorst cites, on p. 1039, n. 1, half a sentence from Van Nooten, *op. cit.*, p. 33: "... nor is it possible to prove that Piṅgala's work existed before the third century A.D." The other half of the sentence, held back by Bronkhorst, gives a totally different emphasis. Therefore I cite the full sentence: "It is not possible, on objective grounds, to decide whether Piṅgala's treatise preceded or followed Pāṇini, nor is it possible to prove that Piṅgala's work existed before the third century A.D."

24. In the introduction to his edition of the *Chandoviciti*, Schlingloff argues persuasively that the so-called classical metres belong actually to a pre-classical period when variation in metres was the most predominant means of poetic ornamentation as against the *alaṃkāras* of the later times and that classical poetry itself employs only a very small fraction of the large repertoire of metres described in works like the *Chandaḥsūtra*.
25. Cf. Van Nooten, *op. cit.*, p. 31: "The Vedic tradition ascribed a great, almost mystical significance to the metres of the sacrificial chants. Careful studies were made not only on the metres of the chant, but also of its language, prosody, proper place and proper time of recitation. The methodology so developed to study and analyze metres became a respected field of study from a very early time onward. In this tradition the earliest comprehensive treatise on Vedic and Sanskrit metres that has been preserved is the *Chandaḥsūtra* by Piṅgala".
26. Cambridge, Mass., 1916, p. 523. See also, Gonda, Jan, *Vedic Literature (Saṃhitās and Brāhmanas)* (A History of Indian Literature, Vol. I, Fasc. 1), Wiesbaden 1975, p. 175 ff.
27. Cf. Bose, D.M., et al (eds.), *A Concise History of Science in India*, New Delhi 1971, pp. 156-57.
28. Van Nooten, whom Professor Bronkhorst cites in order to demolish my chronology, has this to say on the eighth chapter, *op. cit.*, p. 32: "The passage where the binary system developed is in all likelihood part of the original work".