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FROM CHINA TO PARIS: 2000 YEARS TRANSMISSION OF MATHEMATICAL IDEAS

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Rule of Three and its Variations in India

by SREERAMULA RAJESWARA SARMA

trairāśikenaiva yad etad uktaṃ vyāptaṃ svabhedair hariṇeva viśvam.
Just as the universe is pervaded by Hari with His manifestations, even so all that has been taught [in arithmetic] is pervaded by the Rule of Three with its variations.

Bhāskarācārya, *Līlāvatī* (ca. 1150)

Uns haben die Meister der freien Kunst von der Zahl eine Regel gefunden, die heisst Goldene Regel, davon, dass sie so kostbar und nützlich ist gegenüber allen anderen Regeln von gleicher Art, wie Gold alle anderen Metalle übertrifft. Sie wird auch genannt Regeldetri nach welscher Zunge, weil sie aussagt von dreierlei und beschliesst drei Zahlen in sich.

Ulrich Wagner, *Das Bamberger Rechenbuch von 1483*

In the history of transmission of mathematical ideas, the Rule of Three forms an interesting case. It was known in China as early as the first century A.D. Indian texts dwell on it from the fifth century onwards. It was introduced into the Islamic world in about the eighth century. Renaissance Europe hailed it as the Golden Rule. In this paper, I propose to discuss the history of this rule and its variations in India as discussed in the texts in Sanskrit and other languages. I shall dwell on the theoretical deliberations, the types of problems and the mechanical processes by which these are solved by the rule. I conclude the paper with brief notes on the transmission of the rule to the Islamic world and thence to Europe.

In der Geschichte der Übermittlung von mathematischen Ideen ist die Dreisatz-Regel ein interessanter Fall. In China war sie schon im ersten Jahrhundert n. Chr. bekannt. Indische Texte behandeln sie seit dem fünften Jahrhundert. In die islamische Welt wurde sie um das achte Jahrhundert eingeführt. Zur Zeit der Renaissance wurde sie in Europa als „Goldene Regel" bekannt. Die Geschichte dieser Regel und ihrer Varianten werden im folgenden Aufsatz behandelt. Der Aufsatz erklärt die theoretischen Überlegungen, die Arten von Aufgaben, die mit dieser Regel gelöst werden können, die mechanischen Vorgänge, die bei der Lösung der Aufgaben zu benutzen sind, und erörtert dann kurz die Verbreitung der Regel in der islamischen Welt und in Europa.

Introduction

In the history of transmission of mathematical ideas, the Rule of Three forms an interesting case. It was known in China as early as the first century A.D. Indian texts dwell on it from the fifth century onwards but a rudimentary form of the rule was available much earlier. It was introduced into the Islamic world in about the eighth century. Europe hailed it as the Golden Rule. The importance of the rule lies not so much in the subtlety of its theory as in the simple process of solving problems. This process consists of writing down the three given terms in a linear sequence ($A \rightarrow B \rightarrow C$) and then, proceeding in the reverse direction, multiplying the last term with the middle term and dividing their product by the first term ($C \times B \div A$). With this rule one can easily solve several types of problems even without a knowledge of the general theory of proportion.

The writers in Sanskrit, however, were well aware of the theory. Commenting on the rule given by Āryabhaṭa, Bhāskara I notes that this rule encompasses Rules of Five, Seven and others because these are special cases of the Rule of Three itself. Bhāskara II even declares that the Rule of Three pervades the whole field of arithmetic with its many variations just as Viṣṇu pervades the entire universe through his countless manifestations. Leaving aside the poetic hyperbole, there is no doubt that the mechanical methods provided by the Rule of Three and its variations offer quick solutions to nearly all problems concerning commercial transactions.

This mechanical rule held sway over large parts of the world for nearly two thousand years. If the price of A things is B coins, the issue of determining the price of C things must have been one of the earliest exercises in the realm of computation. Problems of this nature being fundamental to every commercial transaction, their solution must have been available in all early civilisations. As Tropfke observes, "the logic on which the Rule of Three is based must belong to the earliest realisations of the counting man" [Tropfke 1930: 187]. However, the earliest records concerning the theoretical deliberations about the problems and their solution emanate from China and India, although their mutual relationship awaits further investigation.

Early History of the Rule of Three

In India the Rule of Three was first mentioned by Āryabhaṭa I in his *Āryabhaṭīya* (A.D. 499):

> Now having multiplied the quantity known as fruit (*phala-rāśi*) pertaining to the Rule of Three (*trairāśika*) by the quantity known as requisition (*icchā-rāśi*), the obtained result (*labdha*) should be divided by the argument (*pramāṇa*). [What is obtained] from this [operation] is the fruit corresponding to the requisition (*icchā-phala*).[1]

Here Āryabhaṭa not only gives the name *Trairāśika* (that which consists of three numerical quantities or terms) for the Rule of Three, but mentions as well the technical terms for the four numerical quantities involved (*pramāṇa*, *phala-rāśi*,

1 [*Āryabhaṭīya*, *Gaṇitapāda* 26]:
trairāśikaphalarāśiṃ tam athecchārāśinā hataṃ kṛtvā /
labdhaṃ pramāṇabhajitaṃ tasmād icchāphalam idam syāt //

icchā-rāśi, *icchā-phala*) and gives the formula for solving the problem. Subsequent writers, notably Brahmagupta in his *Brāhmasphuṭasiddhānta* (A.D. 628) and Bhāskara I in his commentary (A.D. 629) on the *Āryabhaṭīya* elaborate upon this brief statement by Āryabhaṭa, but employ the same terminology, albeit with slight modifications. It is on the basis of the writings of these mathematicians that histories of mathematics generally trace the origin of the Rule of Three to India.[2] The brief manner in which Āryabhaṭa presents the rule in his work implies that he is referring to an already well known rule which he is restating here in order to employ it in astronomical computations. Therefore, it is tempting to look for the antecedents for Āryabhaṭa's rule.[3]

Kuppanna Sastry sees the first mention of the Rule of Three in the following verse of the *Vedāṅgajyotiṣa*, "The known result is to be multiplied by the quantity for which the result is wanted, and divided by the quantity for which the known result is given."[4] Sastry goes on to say:

> The instruction is concise and looks like an aphorism. There are four items in a proportion, three known and one unknown, which is obtained from the three known. Hence the rule to get this is called the "Rule of Three." The four items are: (a) If for so much quantity, (b) so much result is got, (c) for this much quantity given now, (d) how much is the result that will be got? The first two are called *jñāta-rāśis* and the next two are called *jñeya-rāśis*. The application of the rule is: Take the known result, i.e. (b), multiply it with the quantity (c) for which the result is to be known, and divide by the quantity (a) for which the result is given; thus the result to be known, i.e. (d), is got.

It is obvious that we have here a rudimentary form of the Rule of Three and that the rule was needed for the computations envisaged in the text. Although Indians developed special terminology for the Rule of Three in later times, the general terms used here, *jñāna⟨ta⟩rāśi* (the quantity that is known or given) and *jñeya-rāśi* (the quantity that is to be known), are also frequently employed in later times.[5] Indeed, it is conceivable that the term *jñāna* gave rise to the later term *pramāṇa*. However, the date of this text, available in two recensions, is uncertain. Kuppanna Sastry himself would like to place the composition of the text in the period between 1370–1150 B.C.; others assign it to 500 B.C. In either case, Āryabhaṭa's rule appears to have a long prehistory in India.

2 Thus, D. E. Smith, "The mercantile Rule of Three seems to have originated among Hindus. It was called by this name by Brahmagupta (ca. 628) and Bhāskara (ca. 1150), and the name is also found among the Arab and medieval Latin writers" [Smith 1925: 483].

3 The word *rāśi* occurs in [*Chāndogya Upaniṣad* 7.1.2] as the name of a *vidyā* (subject of study) and in [*Ṭhānaṃga Sutta*: 747] as the title of one of the topics of mathematics. There is no reason to suppose that either of these occurrences denotes the Rule of Three. The former might mean mathematics in general and the latter "heap," "group," or "set."

4 Rk-recension 24; Yajus-recension 42: *jñeyarāśi-gatābhystā⟨taṃ⟩ vibhajet jñāna⟨ta⟩rāśinā* [*Vedāṅga Jyotiṣa*: 40–41].

5 Cf. the anonymous commentary on the *Pāṭīgaṇita*, [*Bakhshālī*].

However, Joseph Needham observes that the "Rule of Three, though generally attributed to India, is found in the Han *Chiu Chang*, earlier than in any Sanskrit text. Noteworthy is the fact that the technical term for the numerator is the same in both languages — *shih* and *phala*, both meaning "fruit." So also for the denominator, *fa* and *pramāṇa*, both representing standard unit measures of length" [Needham 1959: 146]. Needham goes on to add that "Even the third known term in the relationship can be identified in the two languages. For *icchā*, 'wish, or requisition' reflects *so chhiu lü*, i.e. ratio, number sought for" [Needham 1959: 146, note i].

In a recent article, N. L. Maiti draws attention to the passage of the *Vedāṅga Jyotiṣa*, in order to counter Needham's claim of Chinese priority. Maiti also disputes Needham's linguistic equation *fa* = *pramāṇa; shih* = *phala; so chhiu lü* = *icchā*. Finally, he tries to clinch the issue by citing [Maiti 1996: 7] the view of a Chinese scholar from Singapore, Lam Lay-Yong: "The rule of three which originated among the Hindus is a device used by oriental merchants to secure results to certain numerical problems" [Lam 1977: 329].

I am not competent to judge in favour or against Needham's linguistic equation, except to say that the word *pramāṇa* does not represent "standard unit measures of length" as words like *aṅgula* or *hasta* do; it merely means, among other things, "size" or "measure." I am sure there are others who can determine the precise meaning of the Chinese *fa* and decide whether the two sets of terminology have the same connotation. But there is no denying the fact that the Rule of Three had an important place in Chinese mathematics as well. Even if the verse from the *Vedāṅga Jyotiṣa* alludes in a rudimentary form to the Rule of Three and thus testifies to the existence of the rule in the centuries before the Christian era, there is nothing to prevent the knowledge of the Chinese *Jiu Zhang* (i.e., Needham's *Chiu Chang*, or *Nine Chapters*) to travel to India in the early centuries of the Christian era and to give impetus to the development of the Rule in India.[6]

Development of the Rule in India

We have seen that the Rule of Three occurs in a rudimentary form in the *Vedāṅga Jyotiṣa* towards 500 B.C. and about a thousand years later it appears in a fully developed form with all the technical terminology in the *Āryabhaṭīya* of Āryabhaṭa. In his commentary on the *Āryabhaṭīya*, Bhāskara I dwells at length on the full implication of the rule given by Āryabhaṭa [*Āryabhaṭīya-Bhāskara*: 116–122]. According to him, Āryabhaṭa's rule encompasses Rules of Five, Seven, etc. This point will be discussed below in detail. Bhāskara also cites a stanza, which states:

6 If India received impetus from China in this process of development and then transmitted an elaborate system to the Middle East and Europe, then this would testify to both the receptivity and creativity in mathematical thought in India. It would, however, be nice if the transmission could be mapped in detail.

> In solving problems connected with the Rule of Three, when the numbers are written down (*sthāpanā*), the wise should know that the two like quantities should be set down in the first and in the last places, and the unlike quantity in the middle [*Āryabhaṭīya-Bhāskara*: 117].[7]

This stanza is preceded by the expression *uktam ca*, "It has also been said." This suggests that the verse was composed by somebody else before Bhāskara's time. This is an important citation, not for the method of solving, but for the method of writing down the numbers. In his rule, Āryabhaṭa did not explain how to set down the three given numerical quantities. This anonymous verse provides therefore the first extant statement that the three quantities should be set down in a certain sequence, viz. $A \rightarrow B \rightarrow C$, and then be worked out in a certain other sequence: $C \times B \div A$. Logically, B should first be divided to get the price of one item (i.e., the rate) and then, the quotient should be multiplied by the requisition ($B \div A \times C$) to obtain the price of the required number of items. But division may produce fractions and operating with them is more difficult than working with integers. Therefore this anonymous verse prescribes multiplication first and division next, and this procedure is followed by all subsequent writers. Besides increasing the chances of computing with integers, this procedure has the added advantage of mechanical neatness in execution: a forward motion from left to right while setting down the quantities and a contrary motion from the right to the left while working out the problem:

forward motion in setting down, viz.	$A \rightarrow B \rightarrow C$
backward motion in computation, viz.	$A \leftarrow B \leftarrow C$.

This anonymous statement is authenticated by Brahmagupta, who formally restates the sequence in these words:

> In the Rule of Three, argument, fruit and requisition [are the names of terms]: the first and last terms must be similar. Requisition, multiplied by the fruit, and divided by the argument is the result.[8]

Brahmagupta is also the first to state that in the Inverse Rule of Three, the direction of the operation will be the reverse (of what it was in the direct Rule of Three). Thus here, the direction of the setting down and computation will be the same as shown below.

Inverse Rule of Three:	setting	$A \rightarrow B \rightarrow C$	
	computation	$A \rightarrow B \rightarrow C$	$(= A \times B \div C)$.

7 [*Āryabhaṭīya-Bhāskara*: 117]:
ādyantayos tu sadṛśau vijñeyau sthāpanāsu rāśīnām /
asadṛśarāśir madhye trairāśikasādhanāya budhaiḥ //
In later times, the terms were also called *ādi* / *ādya* (first), *madhya* (middle) and *antya* (last).

8 [*Brāhmasphuṭasiddhānta* 12.10]:
trairāśike pramāṇa-phalam-icchādyantayoḥ sadṛśarāśiḥ /
icchā phalena guṇitā pramāṇabhaktā phalaṃ bhavati // 12.10 //

Brahmagupta's formulation of the Rule of Three became the model for the subsequent writers who all underscore that the three given terms should be set down in such a way that the first and last be of like denomination and the middle one be of a different kind. Thus in his *Pāṭīgaṇita*, Śrīdhara (ca. A.D. 750) reiterates these points:

> In [solving problems on] the Rule of Three, the argument (*pramāṇa*) and the requisition (*icchā*), which are of the same denomination, should be set down in the first and last places; the fruit (*phala*), which is of a different denomination, should be set down in the middle. [This having been done], that [middle quantity] multiplied by the last quantity should be divided by the first quantity [*Pāṭīgaṇita*, translation: 23].[9]

An anonymous commentary on the *Pāṭīgaṇita*, composed in Kashmir in the ninth or tenth century, has a valuable discussion:

> All this is *trairāśika*. In this method, in the first and last positions, set down the quantities which are of the same class or denomination (*jāti*). Furthermore, in the first place, set down the quantity which is the argument, then [in the last position] the quantity which is the requisition and between these two the fruit which belongs to a different denomination. Having done this, the fruit is multiplied by the last term, namely the quantity called requisition and then divided by the first term, namely the quantity called argument. Thus the desired (*jijñāsita*) result is obtained. Here *jāti* refers to the denomination connected with the commercial transactions. Being of the same denomination or of different denominations is to be understood in this sense and not in the sense of caste as applicable to Brahmins etc. ... If the argument is a commodity (*paṇanīya*) and the requisition is also a commodity, they are of the same class. Then the middle term will be the price (*mūlya*), because of the mutual dependence (*parasparāpekṣitatva*) of the commodity (*paṇya*) and price. Likewise, if the first and last terms are the prices, then the commodity will be the middle term. If the two [first and last terms] refer to artisans of the same denomination, then their wages (*bhṛti*) is the middle term. Or the amount of work done by the artisans is of a different denomination. Therefore [a statement related to their amount of work] is the middle term. While the first and the last terms are of like denomination, the middle term is of the same denomination as the quantity which is desired to be known [*Pāṭīgaṇita*: 37].

Other mathematicians, for example, Mahāvīra (ca. A.D. 850) and the second Āryabhaṭa (ca. A.D. 950), do not add much to the theory of the Rule of Three.[10] The celebrated Bhāskara II also follows suit in his *Līlāvatī*, but with a certain economy of expression:

> The argument and requisition are of like denomination; they are to be set down in the first and the last places. The fruit, which is of a different denomination, is set down in the middle. That [middle term], being multiplied by the requisition and divided by the first term, gives the fruit of the requisition. The operation is reversed in the inverse (*viloma*) method.[11]

Bhāskara II deserves special notice because, towards the close of his *Līlāvatī*, he declares that nearly the entire arithmetic is based on the Rule of Three and that most of the topics are but variations of this Rule of Three:

> Just as the universe is pervaded by Hari with His manifestations, even so all that has been taught [in arithmetic] is pervaded by the Rule of Three with its variations.[12]

He goes on to elaborate this further in the following words:

> As Lord Śrī Nārāyaṇa, who relieves the sufferings of birth and death, who is the sole primary cause of the creation of the universe, pervades this universe through His own manifestations as worlds, paradises, mountains, rivers, gods, men, demons, etc., so does the Rule of Three pervade the whole of the science of calculation. ... Whatever is computed whether in algebra or in this [arithmetic] by means of multiplication and division may be comprehended by the sagacious learned as the Rule of Three. What has been composed by the sages through the multifarious methods and operations such as miscellaneous rules, etc., teaching its easy variations, is simply with the object of increasing the comprehension of duller intellects like ourselves [*Līlāvatī*, auto-commentary on 240, and verse 241; trans. Datta & Singh 1962, I: 209].

Again, in the *Siddhāntaśiromaṇi*, Bhāskara II reiterates that arithmetic is basically the Rule of Three only and goes on to say that:

> Leaving squaring, square-root, cubing and cube-root, whatever is calculated is certainly a variation of the Rule of Three, nothing else. For increasing the comprehension of duller intellects like ours, what has been written in various ways by the learned sages ..., has become arithmetic [*Siddhāntaśiromaṇi*, Golādhyāya, Praśnādhyāya, 3–4; trans. Datta & Singh 1962, I: 210].

Rule of Three and Proportion

Were these writers aware that the Rule of Three is based on proportion? D. E. Smith observes that "Proportion was thus concealed in the form of an arbitrary rule, and the fundamental connection between the two did not attract much notice until, in the Renaissance period, mathematicians began to give some attention to commercial arithmetic" [Smith 1925: 484]. This statement is not based on valid grounds.

9 [*Pāṭīgaṇita*: 37 (Rule 43)]:
ādyantayos trirāśāv abhinnajātī pramāṇam icchā ca /
phalam anyajāti madhye tad antyaguṇam ādinā vibhajet //
The translation is by K. S. Shukla [*Pāṭīgaṇita*: 23].

10 The second Āryabhaṭa, however, uses two different terms: *māna* for the first term and *vinimaya* for the middle term; cf. [*Mahāsiddhānta* 15.24–15.25].

11 [*Līlāvatī*: 73]:
pramāṇam icchā ca samānajātī ādyantayoḥ staḥ phalam anyajātiḥ /
madhye tadicchāhatam ādihṛt syād icchāphalaṃ vystavidhir vilome //

12 [*Līlāvatī*: 239]:
trairāśikenaiva yad etad uktaṃ vyāptaṃ svabhedair hariṇeva viśvam //

That the Rule of Three (*trairāśika*) was a case of proportion (*anupāta*) is well known, even if the formula does not show it in the manner in which the West is accustomed to see it ($a : b :: c : d$).[13] This will be evident from Bhāskara's comments on Āryabhaṭa's rule: "In this rule, Āryabhaṭa described only the fundamentals of proportion (*anupāta*). All others such as the Rule of Five etc. follow from that fundamental rule of proportion" [*Āryabhaṭīya-Bhāskara*: 116].

Commenting on the same passage, Sūryadeva Yajvan states:

> Here we have the logical proposition (*vācoyukti*) — if by so many coins so many things are obtained, by so many coins how many things will be obtained? Here the first quantity is called *pramāṇa*; the second quantity is *phala* and the third is *icchā*. With these three quantities, the fourth is determined.[14]

Indeed, while solving the problems posed in the texts, the commentaries often state such propositions in order to show the logic behind the various steps of computation.[15] Bhaṭṭotpala clearly states that the proportions are called the mathematics of the Rule of Three.[16]

Furthermore, the Rule of Three is often used as a means of verification in solving other problems. Thus for instance, Bhāskara I, in his commentary on *Āryabhaṭīya* 2.25 which contains a problem on interest, employs the Rule of Five for verification.[17] The *Bakhshālī Manuscript* frequently employs the Rule of Three for verification with the words *pratyaya(s) trairāśikena* [*Bakhshālī*: 157, 159 *et passim*]. So does Gaṇeśa in his commentary *Buddhivilāsinī* on the *Līlāvatī* [*Buddhivilāsinī*: 83 *et passim*]. Already in the ninth century, Govindasvāmin attempts to relate the rule to the science of logic. He sees it as a case of inference. Just as there is an invariable concomitance between smoke and fire, so it is between the argument and the fruit. Just as this invariable concomitance allows one to infer that there is fire on the mountain because there is smoke, so does the relation between the argument and fruit allow us to compute the fruit of requisition from the requisition.[18]

13 But is this not so with every formula, that it conceals the logic behind it? See [Datta and Singh, I: 217]. See also [Juschkewitsch 1964: 119–120]: "Die Erläuterung der Regeln war in den indischen Werken formaler Art, doch hatten die indischen Mathematiker unzweifelhaft Verständnis für deren gemeinsame Grundlage und deren Querverbindungen."

14 [*Āryabhaṭīya-Sūryadeva*: 65]:

> *atra hīyam vācoyuktiḥ etāvadbhir etāvanti labhyante etāvadbhiḥ kiyantīti / tatra prathamaḥ pramāṇarāśiḥ dvitīyaḥ phalarāśiḥ tṛtīya icchārāśiḥ / taiś caturtho rāśiḥ sādhyate /*

15 Cf. [Nīlakaṇṭha, III: 49]: *trairāśika-vācoyuktiś caivam.* Throughout this commentary, there are such statements of proportion.

16 In his commentary on Varāhamihira's *Laghujātaka* 6.2 he states: *anupātās trairāśika-gaṇitam abhidhīyate*, as cited in the *Petersburger Wörterbuch*, s.v. *trairāśika.*

17 [*Āryabhaṭīya-Bhāskara*: 114–115]: *pratyayakaraṇam pañcarāśikena.*

18 His views are cited in [*Kriyākramakarī*: 179–183]. For a lucid exposition of these views, see [Hayashi 2000: 210–226]. Here, Hayashi wishes to render the term *trairāśika* as "three-quantity operation."

Applications of the Rule of Three

After thus discussing the views of various mathematicians, we may now look at the areas where this rule is applied, in other words, the problems concerning the Rule of Three. It is obvious that Āryabhaṭa's main purpose in enunciating the Rule of Three is to employ it in astronomical computations. One of these is the computation of the mean position of a planet from the number of its revolutions in a *Kalpa* of 4,320,000,000 years.[19] Many of the problems of spherical astronomy are also solved by the application of the Rule of Three to the similar triangles called *akṣakṣetra*, "latitude-triangles" [*Āryabhaṭīya-Trans.*: 130–132]. Likewise, the Rule of Three forms the basis for trigonometrical ratios [Gupta 1997: 73–87]. Therefore, Nīlakaṇṭha Somasutvan (b. A.D. 1444) declares, in his commentary on the *Āryabhaṭīya*, that the entire mathematical astronomy (*graha-gaṇita*) is pervaded by two fundamental laws: by the law of [the relation between] the base, perpendicular and hypotenuse [in a right-angled triangle] and by the Rule of Three.[20]

However, the Rule of Three became well known outside India for its application in the so-called commercial problems in arithmetic.[21] It is Bhāskara I who, in the course of his commentary on Āryabhaṭa's rule, gives various examples of such commercial problems for the first time [*Āryabhaṭīya-Bhāskara*: 116–122]. Since Brahmagupta did not give any examples of his own, these given by his contemporary Bhāskara are the earliest. Therefore, it is necessary to look more closely at these earliest examples provided by Bhāskara and his methods of solving them.

Bhāskara gives altogether seven problems or examples (*uddeśaka*): (i) price and quantity of sandalwood; (ii) price and weight of ginger (the problem has fractions and illustrates Āryabhaṭa's rule for fractions); (iii) price and quantity of musk also with fractions; (iv) time taken by a snake in entering a hole; (v) mixed quantities (*miśrakarāśis*); (vi) partnership (*prakṣepakaraṇa*); (vii) partnership expressed as fractions (*bhinna*). Of these, (i) is a case of simple proportion; (iii) is similar to (ii). We shall examine how Bhāskara solves the other five problems. We shall also see how some of these became new topics in later times.

Price and weight of ginger. Bhāskara's second problem involves fractions:

> Ginger, of 1 *bhāra* weight, was sold at 10 and 1/5 coins. What is the price of ginger of 100 and 1/2 *palas*? [*Āryabhaṭīya-Bhāskara*: 117]

Since 1 *bhāra* equals 2000 *palas*, the three terms are set down as follows. This process of setting down the given terms in the prescribed order is called *nyāsa*:

19 Cf. [Nīlakaṇṭha, III: 1]: *ahargaṇāt trairāśikena madhyamam ānīya sphuṭīkriyate.*

20 [Nīlakaṇṭha, I: 100]: *bhujākoṭikarṇanyāyena trairāśikanyāyena cobhābhyāṃ sakalaṃ graha-gaṇitaṃ vyāptam.* See also [Nīlakaṇṭha, I: 19].

21 Even non-mathematical texts in India stress the importance of the Rule of Three for solving commercial problems. Thus King Someśvara lays down in his *Mānasollāsa* (A.D. 1129) that the accountant in the royal treasury should be thoroughly versed in multiplication and division [with fractions] and the method of [applying] the Rule of Three (*trairāśika-vidhāna*); cf. [*Mānasollāsa* 2.2.124 (p. 40)]; see also [*id.* 2.2.113–2.2.116 (p. 39)] where the rules of three, five, seven and nine terms are discussed.

2000	10	100
	1	1
	5	2.

It may be noted that this is how mixed fractions are written in India: integer, numerator and denominator one below the other without any lines separating them. Bhāskara explains the next step thus:

> After assimilating the integers with the related fractions by reducing them to a common denominator (*savarṇita*), the three quantities are set down once more in the following manner:
>
> | | | |
> |---|---|---|
> | 2000 | 51 | 201 |
> | | 5 | 2. |
>
> Then, following Āryabhaṭa's rule, the denominators of the two multipliers are transposed to the divisor. Thus by the two denominators 5 and 2, the divisor (2000) is multiplied, it becomes 20000. The product of 201 and 51 is 10251. This is divided by 20000. The result is 10251/20000 coins ...

Snake entering a hole. Bhāskara's fourth problem is the following:

> A snake, twenty cubits long, enters a hole at the rate of half *aṅgula* per *muhūrta* and it comes out by one-fifth *aṅgula*. How many days does it take to enter the hole? Setting down: the snake, 480 *aṅgula*s long, goes in by 1/2 *aṅgula*, comes out by 1/5 *aṅgula* in 1 *muhūrta*. Thus its rate of entering per *muhūrta* is half *aṅgula* diminished by one-fifth *aṅgula*. So subtract one-fifth from one-half, and set the terms down:
>
> rate of entry 3/10 *aṅgula* in *muhūrta* 1, snake's length in *aṅgula*s 480
>
> [*Āryabhaṭīya-Bhāskara*: 118].

The calculation implied is: $480 \times 1 \times 10/3 = 4800/3 = 1600$ *uhūrtas* $= 1600/30$ days $=$ 53 1/3 days.

Śrīdhara treats this as an independent topic called *gati-nivṛtti* (forward and backward motion) and provides a separate rule for solving it, "On subtracting the backward motion per day from the forward motion per day, the remainder is the (true) motion per day" [*Pāṭīgaṇita*: Rule 44ab (p. 41)]. Then this net daily motion is treated as the argument.

Why do we need this new rule? Can we not solve this as Bhāskara did above? The commentary on the *Pāṭīgaṇita* says that it is for the convenience of pupils (*śiṣyahitārthatvāt*):

> Subtract the backward motion per day from the daily forward motion. The remainder will be the net daily motion. Take that as argument, 1 day as the fruit, and demand the time taken for the distance to be traversed. Thus there are three Rules of Three: the first to derive the daily forward motion, the second to get the daily backward motion, and the third to compute the time taken for the distance to be traversed. These can of course be worked out according to the previously given general rule. Even so, it is convenient for the pupils to have a new rule which provides for the argument and fruit to be used for deriving the time taken for the distance to be traversed. Being ignorant, the pupils might try to compute the time for the distance to be traversed, without first subtracting the rate of backward motion, and then compute the amount of backward motion for the above period of time, and then take their difference; but then that would not be correct [*Pāṭīgaṇita*: 41].

Śrīdhara gives two examples, the second of which reads as follows:

> A man earns 1/2 less 8 silver pieces in 1 and 1/3 days; he spends 1/2 silver per day on his food. In how many days will he become the lord of one hundred pieces of silver (*śateśvara*)?[22]

Mahāvīra's *Gaṇitasārasaṃgraha* treats this topic rather elaborately with one rule and several lengthy examples. His rule is as follows:

> Write down the net daily movement, as derived from the difference of [the given rates of] forward and backward movements, each [of these rates] being [first] divided by its own [specified] time; and then in relation to this (net daily movement), carry out the operation of the rule-of-three [*Gaṇitasārasaṃgraha* 5.23].

Of the various examples given by Mahāvīra, the following may be cited:

> A well completely filled with water is 10 *daṇḍas* (= 960 *aṅgulas*) in depth. A lotus sprouting therein grows from the bottom at the rate of 2 1/2 *aṅgulas* in a day and half; the water thereof flows out through a pump at the rate of 2 1/20 *aṅgulas* [of the depth of the well] in 1 1/2 days; while 1 1/5 *aṅgula*s water are lost in a day by evaporation; a tortoise below pulls down 5 1/4 *aṅgulas* of the stalk of the lotus plant in 3 1/2 days. By what time will the lotus be at the same level with the water in the well? [*Gaṇitasārasaṃgraha* 5.28–30]

These recall the lion in the pit problem posed by Leonardo of Pisa in the *Liber Abbaci:*

> The pit is 50 feet deep. The lion climbs up 1/7 of a foot each day and then falls back 1/9 of a foot each night. How long will it take him to climb out of the pit?

However, Leonardo does not employ the Indian method for solving this problem, instead he "uses a version of false position. He assumes the answer to be 63 days, since 63 is divisible by both 7 and 9. Thus in 63 days the lion will climb up 9 feet and fall down 7, for a net gain of 2 feet. By proportionality, then, to climb 50 feet, the lion will take 1575 days" [Katz 1993: 283–284].[23]

Mixed quantities. Going back to Bhāskara, his fifth problem concerns "mixed quantities" (*miśrarāśis*). He adds that here also the same principle of proportion (*anupātābhāym*) applies. The problem is as follows:

22 Note the expression *śateśvara*, "lord of one hundred pieces of money," apparently the eighth-century version of "millionaire."

23 Katz adds: "By the way, Leonardo's answer is incorrect. At the end of 1571 days the lion will be only 8/63 of a foot from the top. On the next day he will reach the top."

Eight bulls are tame and three are untame: thus it has been said about [a group of] bulls. Then in 1001, how many are tame and how many otherwise?
Statement: tame 8, to be tamed 3, total tame and untame 1001.
Now we set down the terms for the Rule of Three thus:
tame and untame 11, tame 8, group of all 1001.
Here the proposition (*vācoyukti*) is: if out of eleven bulls which are tame and untame, eight are tame, then out of 1001 how many are tame?
The result is (1001 × 8 / 11 =) 728. Subtract this number from the total to get (1001 – 728 =) 273 untame bulls [*Āryabhaṭīya-Bhāskara*: 118–119].

This class of problems is styled *miśraka* "mixed" because one of the three terms is the sum of two different quantities; in the present case, the numbers of tame and untame bulls; and the problem involves finding these two numbers separately. However, the only mixed quantity which has any practical relevance would be the sum of the principal and the interest accrued at the end of a given period.

Partnership. The sixth problem, related to *prakṣepa-karaṇa* "partnership", is as follows:

Five merchants in partnership with capitals amounting to 1 increased by 1 each time, earned a profit of 1000. Tell the share of each person.
Statement: capitals 1, 2, 3, 4, 5 / profit 1000 /
Computation (*karaṇa*) — one should formulate a series of statements of proportion, such as: From the sum of the capitals invested of 15, there accrues a profit of 1000. Then from the capital of 1 how much profit accrues; from the capital of 2 how much profit accrues, and so on.
For the first proposition, the answer is 66 2/3; second 133 1/3; third 200; fourth 266 2/3; fifth 333 1/3 [*Āryabhaṭīya-Bhāskara*: 119].

The last example also deals with partnership, but here the shares of investment are expressed in fractions (*bhinna*):

Merchants who invested 1/2, 1/3 and 1/8 earned a profit of 69. What are the individual shares?

Statement:	1	1	1	profit
	2	3	8	

By taking a common denominator for all these fractions, we get 12/24, 8/24 and 3/24. The denominators are of no account and therefore we take only the numerators 12, 8, 3. As in the previous case of partnership, take their sum 23. For this 23 capital invested (*prakṣepa*) the profit is 69, then how much for each individual; by the Rule of Three we get 36, 24, 9 [*Āryabhaṭīya-Bhāskara*: 119].

Bhāskara's examples thus cover the whole range where the direct Rule of Three can be applied. Later mathematicians like Śrīdhara and Mahāvīra created independent topics out of these variations, such as *gati-nivṛtti* "foreward and backward motion," *prakṣepa-karaṇa* "partnership" and several types of *miśraka* "mixed quantities," and formulated separate rules for their solution.

Rules of Five, Seven or More Terms

Another set of important variations of the Rule of Three are the rules for five, seven and more terms, called respectively *Pañcarāśika*, *Saptarāśika*, *Navarāśika*, etc. In his commentary on the *Āryabhaṭīya*, Bhāskara argues that these are just special cases of the Rule of Three:

Here Ācārya Āryabhaṭa had described the Rule of Three only. How are the well-known Rules of Five, etc., to be obtained? I say this: The Ācārya [Master] has described only the fundamentals of *anupāta* (proportion). All others such as the Rule of Five, etc., follow from that fundamental rule of proportion. How? The Rule of Five, etc., consists of combination of the Rule of Three ... In the Rule of Five there are two Rules of Three, in the Rule of Seven, three Rules of Three, in the Rule of Nine, four Rules of Three, and so on [*Āryabhaṭīya-Bhāskara*: 116; trans. Datta and Singh 1962, I: 211].

Bhāskara gives the following example for the Rule of Five:

On 100 the interest in 1 month is 5. Then what is the interest on 20 for 6 months? Tell if you understood [Ārya]bhaṭa's mathematics [*Āryabhaṭīya-Bhāskara*: 119–120].

and explains the solution in the following words:

Computation (*karaṇa*). First Rule of Three: 100, 5, 20; result 1 silver coin. Second Rule of Three: if the interest on 20 is 1, how much in 6 months? result 6 silver coins.

Bhāskara goes on to add:

If the same computation is performed in one go, it becomes the Rule of Five. There also, we have two numerical quantities as argument (*pramāṇa-rāśis*), namely (100 and 1); 5 is the fruit; [we ask] how much on 20 in 6 months; thus 20 and 6 come under requisition (*icchārāśi*). As before, quantities of requisition (*icchārāśis*) are multiplied by the quantity of fruit (*phalarāśi*) and divided by the two quantities of argument (*pramāṇa-rāśis*). We get the same result as before: (20 × 6) (5) ÷ (100 × 1) = 6. This is just the Rule of Three set down in two different ways. As before, the denominators of fractions also are mutually transposed from division and multiplication.[24]

Bhāskara's second example reads thus:

If on 20 and 1/2 in 1 and 1/5 month the interest is 1 and 1/3 silver, on 1/4 less 7 what is the interest in 6 and 1/10 months? [*Āryabhaṭīya-Bhāskara*: 121]

Bhāskara does not state all the steps, but he would probably proceed as follows:

24 Strangely enough, *Bakhshālī* does not treat the Rule of Five as a separate entity but employs two successive Rules of Three for solving problems with five terms; cf. [*Bakhshālī*: example 38 (X-21), p. 427]; it does the same, as will be shown below, also with the Inverse Rule of Three.

After converting all the quanties into regular fractions, set down the terms in a row:

41/2, 6/5, 4/3, 27/4, 61/10 I

Multiply the fruit with the two quantities of requisition and divide by the two quantities of argument:

(4/3 × 27/4 × 61/10) ÷ (41/2 × 6/5) II

Now the denominators of one group become the multipliers of the numerators of the other. Transpose these accordingly:

(4 × 27 × 61 × 2 × 5) ÷ (41 × 6 × 3 × 4 × 10) III

The result then is 2 171/1226.

Brahmagupta condenses this process and tells us how to reach step III. This is achieved by: (i) arranging the given data in two columns, the first column containing the data concerning the argument, the second containing that concerning the requisition; (ii) transposing the two fruits; (iii) transposing the denominators. Then the product of the column with more numerous terms is divided by that of the column with the lesser number of terms. His rule reads as follows:

> In the case of uneven terms, from three up to eleven, transpose the fruit on both sides. The product of the more numerous terms on one side, divided by that of the fewer terms on the other, provides the answer. In all the fractions, transpose the denominators, in a like manner, on both sides.[25]

Thus, by reading Bhāskara and Brahmagupta together, we can see the rationale of this method. The texts do not expressly state that the quantities should be set down in two vertical columns. Gaṇeśa, however, says in his commentary on Bhāskara's *Līlāvatī* that the quantities on the argument side should be written one below the other; so also the quantities on the requisition side. This is tantamount to two vertical columns [*Buddhivilāsinī*: 76].

The manuscripts consistently set down the quantities in this manner in two columns; sometimes the quantities are enclosed in boxes with single or double borders. Though the extant manuscripts are not very old, they may be following a

25 [*Brāhmasphuṭasiddhānta* 12.11–12.12]:

trairāśikādiṣu phalam viṣameṣv ekādaśānteṣu //11//
phalasaṅkramaṇam ubhayato bahurāśivadho 'lpavadhahṛto jñeyam /
sakaleṣv evam bhinneṣūbhayataś chedasaṅkramaṇam //12//

Usually, there is only one fruit, which pertains to the argument side. Then why does Brahmagupta ask us to transpose the fruits? Because, there can also be problems where the fruits or the interests of the two sides are given and one is asked to compute the principal or the time on the requisition side. Again, when the fruit is transposed from the first column to the second, the second will have more numerous terms. Then why does he not say: divide the product of the second column by that of the first column? This is usually the case, but there can be instances of more terms in the first column. It must also be kept in mind that terms mean only the numerators and not the denominators.

continuous tradition, just as they do while setting down the terms in the Rule of Three in a horizontal sequence.[26]

Thus two methods seem to be prevalent for setting down the quantities: a horizontal one for the Rule of Three and a vertical double column for Rules of Five and others. However, it appears that the three terms of the *trairāśika* were occasionally written in a vertical column as well. In *Bakhshālī*, there are some 44 problems where the Rule of Three is applied [*Bakhshālī*: 421–429]. The terms are set down in horizontal rows in all the cases, except in one where they are set in a vertical column as shown below [*Bakhshālī*: 270; Maiti 1996: 3–4].

1 *dramma*
100 *trapusa*
1 2

The problem here is, "One hundred pieces of tin are obtained for one *dramma*. How many are obtained for a half [*dramma*]." Note that here the denominations are also set down along with the numerical quantities.

Another case occurs in the recently published mathematical text *Caturacintāmaṇi* of Giridharabhaṭṭa (*fl.* 1587). In an example dealing with Rules of Three, Five and Seven, the terms in the three problems are arranged vertically: the terms of the Rule of Three in a single column, those of Five and Seven in two columns [*Caturacintāmaṇi*, verse 34, p. 142 (text), p. 164 (translation)].

> If six is [obtained] by means of five, what is [obtained] by means of eight? Or else, if [that is obtained] in one month, then what is [obtained] in ten [months]? If the result [is obtained] from three people, then what is [obtained] from five? say separately.

5	month	1	10	people	3	5
6		5	8	month	1	10
8		6			5	8
					6	
Rule of Three	Rule of Five			Rule of Seven		

On the other hand, in his *Rāshikāt al-Hind*, al-Bīrūnī consistently uses vertical double columns for setting down the terms whether they are three or seventeen. For example, he sets down the three terms of the Rule of Three in the following manner:

15	5
	3 ,

26 Again, in some manuscripts, in the cell for the unknown quantity (*jñeya*) a zero is written, just as we write an x today. This zero is a mere symbol. It should not be confused with the quantity zero and used in the multiplication along with the other numbers in the column.

where the argument is 5, fruit is 3 and the requisition 15. He says expressly that the terms are arranged with two mutually intersecting lines. Surely, this arrangement also must have had an Indian prototype. It may be recalled that Brahmagupta prescribes vertical double columns for all rules with odd terms from three up to eleven. Some Indian manuscripts must have followed this custom, which al-Bīrūnī emulated.

Thus, whatever may have been the arrangement employed for the Rule of Three, the arrangement finally adopted for the Rule of Five and others was evolving in the seventh century. Writing in 629, Bhāskara was not fully aware of this; but writing in 628 Brahmagupta prescribes it. Therefore, it is quite certain that Brahmagupta himself must have invented this method of writing down in two columns. Later writers, such as the anonymous commentator on the *Pāṭīgaṇita*, designate these two columns clearly as *pramāṇa-rāśi-pakṣa* (the side containing quantities belonging to the argument) and *icchā-rāśi-pakṣa* (the side containing quantities belonging to the requisition).

But is it necessary to have a new rule and a new arrangement? Why cannot these problems be solved by a series of Rules of Three? To this, the commentary on the *Pāṭīgaṇita* again provides the answer:

> The Rule of Five, etc., can be solved by postulating several successive Rules of Three. But identifying in each case correctly the argument, fruit and requisition, requires certain logical ability, which the pupil might not have. Therefore this new rule.[27]

Besides the economy in time and space, there is another advantage which the commentary repeatedly emphasises while working out the problems. After arranging the terms in two columns, and transposing the fruit and the denominators, common factors in the two columns can be cancelled out more easily. Then computing the products in the two columns becomes that much simpler.

Although these problems may contain any odd number of terms, the writers usually go up to eleven terms only. Al-Bīrūnī states that he encountered in India problems containing more than eleven terms and gives problems containing up to seventeen terms in his treatise [Juschkewitsch 1964: 214]. However, the only example that I have yet found with more than eleven terms occurs in the *Gaṇitalatā*, composed by Vallabha in 1841:

> If 1 house, 5 cubits in width, 16 cubits long, with 2 storeys and 2 inner courts, is available at the rate of 3 *niṣkas* per month, how much money is needed for 4 houses, each 6 cubits wide and 18 cubits long, with 4 storeys and 3 inner courts for 5 months? [*Gaṇitalatā*: *Trairāśikakusuma*, verse 13 (f. 26v)]

The thirteen terms can be arranged in two vertical columns as shown below:

27 [*Pāṭīgaṇita*: 45]: *pañcarāśyādiphalam anekatrairāśika-karma-sādhyam / pramāṇaphalecchayā vyavasthayā cānayanasaraṇiḥ śiṣyasya durjñeya iti sūtrāntarārambhaḥ.*

1	4
5	6
16	18
2	4
2	3
1	5
3	

After transposing the fruit and cancelling out the common factors, what remain are 3, 9, 3, 3 in the right column. Their product 243 is the result.

We do not know much about the state of mathematics in the regional languages of India. While looking for Telugu manuscripts, I found an elaborate classification of the variations of the Rule of Three in one manuscript. Because of the disjointed nature of the manuscript, all the variations are not clear to me. Therefore, I shall not enumerate them now. But what is clear is a simpler method of solving the problems of Rule of Five, etc. Set down all the terms horizontally in a sequence. If there are n terms, take the product of the last $(n + 1)/2$ and divide this by the product of the first $(n - 1)/2$ terms. Thus in the case of the Rule of Five, the product of the last three terms is divided by the product of the first two terms; or in the case of the Rule of Seven, the product of the fourth, fifth, sixth and seventh terms is divided by the product of the first, second, and third terms.[28] This may be illustrated by working out the previous example. First we set down all the thirteen terms in the proper sequence:

<u>/1 / 5 / 16 / 2 / 2 / 1/</u> <u>3 / 4 / 6/ 18 / 4 / 3 / 5/</u>

We divide the product of the last seven terms by the product of the first six terms:

$$(3 \times 4 \times 6 \times 18 \times 4 \times 3 \times 5) \div (1 \times 5 \times 16 \times 2 \times 2 \times 1)$$

This, in effect, is what Bhāskara I seemed to suggest, before Brahmagupta proposed the arrangement of two vertical columns. The Telugu solution then, is the ultimate stage of mechanical solution, especially if there are no fractions. Unfortunately, neither the date nor the author of this text fragment is known.

Barter and other Types of Problems

Barter (*bhāṇḍapratibhāṇḍaka*) is treated as an extension (*atideśa*) of the Rule of Five,[29] so also several other types of problems. Mathematicians enjoyed formulating separate rules for solving these. Brahmagupta provides a special rule for barter:

28 In a manuscript owned by Mantri Gopalakrishna Murthy the rule reads thus in Telugu:
kaḍarāsul avi nālgu kramamuto guṇiyiṃci
dravyamaṃdu beṃci dānikriṃda /
modalu rāsulu mūḍu mudam oppa guṇiyiṃci
pālubuccavaccu saptarāśī //

29 [*Buddivilāsinī*: 83]: *atropapattis trairāśikadvayena /*

> In the barter of commodities, transposition of prices being first terms takes place; and the rest of the process is the same as above directed [*Brāhmasphuṭa-siddhānta* 12.13].

This can be demonstrated with the following example:

> If a hundred of mangoes be purchased for ten *paṇas*, and of pomegranates for eight, how many pomegranates [should be exchanged] for twenty mangoes?
> Statement:

10	8
100	100
20	

> After transposition of prices and also of the fruit,

8	10
100	100
	20

> Result: 25 pomegranates.

Inverse Rule of Three

While the elder Bhāskara merely states that the Inverse Rule of Three (*vyasta-trairāśika* or *viloma-trairāśika*) is the reverse of the direct rule, Brahmagupta is the first to spell out the rule in full, "In the inverse rule of three terms, the product of argument and fruit, being divided by the requisition, is the answer."[30]

The younger Bhāskara lays down where this inverse rule is to be employed:

> When there is diminution of fruit if there be increase of requisition, and increase of fruit if there be diminution of requisition, then the inverse rule of three is [employed]. For instance, when the value of living beings is regulated by their age; and in the case of gold, where the weight and touch are compared; or when heaps are subdivided; let the inverted rule of three terms be [used] [*Līlāvatī*: 77–78; trans. Colebrooke 1973: 34].

Of all these varieties of problems dealt with under the Inverse Rule of Three, those concerning the sale of women became notorious. Al-Bīrūnī mentions the following in his [*India* I: 313], "If the price of a harlot of 15 years be, e.g., 10 denars, how much will it be when she is 40 years old?"

Al-Bīrūnī's source must be similar to the one given by Pṛthūdaka in his commentary on the *Brāhmasphuṭasiddhānta*:

> If a sixteen year old wench, her voice sweet like that of a cuckoo and Saurus crane, dancing and chirping like a peacock, receives 600 coins, what would one of 25 years get? [*Brāhmasphuṭasiddhānta*: 769]

30 [*Brāhmasphuṭasiddhānta* 12.11]: *vyastatrairāśikaphalam icchābhaktaḥ pramānaphala-ghātaḥ*. Śrīdhara also gives a similar definition, substituting, however, argument, etc., by first, last and middle terms [*Pāṭīgaṇita*: Rule 44 cd].

Nārāyaṇa, the author of the *Gaṇitakaumudī*, has a more modest aim; he does not compute the price for the purchase, but only wants to know the fee for one night:

> If a woman of sixteen, with agreeable gestures, wit and coquetry, gets a fee (*bhāṭi*) of ten *niṣkas*, then tell me quickly, how much should the customer give to one of twenty years? [*Gaṇitakaumudī*, I: 49]

As Bhāskara noted, the price of living beings, be they slaves or draught animals, is inversely proportional to their age. But the price does not increase indefinitely as the age decreases. There is the optimum age that receives the maximum price. In the case of female slaves to be employed for manual labour or for sexual enjoyment, sixteen appears to be the optimum age. Gaṇeśa Daivajña notes that, "A woman of sixteen reaches the optimum as regards her body and qualities. Therefore she receives the maximum price" [*Buddhivilāsinī*, Part I: 74].

One of the problems given by Śrīdhara under Inverse Rule of Three has five terms:

> Some quantity of yarn was used in weaving blankets of breadth 3 and length 9, the blankets thus woven are 200. With the same yarn how many blankets can be woven of breadth 2 and length 6 units [*Pāṭīgaṇita*: example 37 (p. 44)].

However, both Śrīdhara and the commentator treat this as a case of the Inverse Rule of Three; they first calculate the area of each type of carpet and put down these areas under argument and requisition. Thus states the commentary:

> Here by the multiplication of the breadth and length, the area is known. Thus the product of 3 and 9 is 27. The product of 2 and 6 is 12. The area measure of 27 is the given quantity (*jñātameyasaṃkhyā*). Therefore it is the argument. The quantity 12 is the requisition by inversion. The known number is the middle term. Statement: 27 / 200 / 12. By proceeding according to the given rule, the result obtained is 450.[31]

Mahāvīra envisages, besides the Inverse Rule of Three (*vyasta-trairāśika*), also the Inverse Rule of Five (*vyasta-pañca-rāśika*), the Inverse Rule of Seven (*vyasta-sapta-rāśika*), and the Inverse Rule of Nine (*vyasta-nava-rāśika*), but does not explain how these have to be worked out. Probably he would solve these in the same way as Inverse Rule of Three. Since the additional terms are attributes to either the argument or to the requisition and are directly proportional to the same, the various terms under the argument are multiplied and their product is treated as the argument in the Inverse Rule of Three; the same is done with the terms under the requisition; so that finally, there are only three terms, which are then dealt with according to the Inverse Rule of Three. We consider his problem on the Inverse Rule of Seven:

31 [*Pāṭīgaṇita*, Commentary: 44]:
atra viṣkambhāyāmayor vadhān mānaparicchedāḥ / tena trikanavavadhaḥ sapta-viṃśatiḥ dvikaṣaṭkavadho dvādaśa / saptaviṃśatimānasya jñātameyasaṃkhyatvād pramāṇatvaṃ dvādaśakasya viparyāsād icchātvaṃ jñātā samkhyā madhyamo rāśiḥ /

> Out of a gigantic ruby, measuring 4, 9, 8 cubits respectively in length, breadth and height, how many icons can be carved of Tīrthaṅkaras, each measuring 2, 6, 1 cubits respectively in length, breadth and height? [*Gaṇitasārasaṃgraha* 5.21]

Probably, Mahāvīra would set down the numbers thus:

4, 9, 8	1	2, 6, 1

Take the product of each group and proceed as in the Inverse Rule of Three:

$$(4 \times 9 \times 8) \times 1 \div (2 \times 6 \times 1) = 288 \div 12 = 24.$$

We have seen that Rules of Five, Seven, etc., do not appear in the *Bakhshālī Manuscript*. This text does not seem to be aware of the Inverse Rule of Three either. The two problems that are available in this (somewhat mutilated) work are solved by employing the Rule of Three successively two times [*Bakhshālī*: 426–427, examples 35–38]. Let us take the following problem, "If one person can live on eight *drammas* for forty-two days, then how long can seventy persons live on the same amount of money?"

In the method of the Inverse Rule of Three, the three terms are 1 / 42 / 70 and the result is $1 \times 42 \div 70 = 3/5$ days. The *Bakhshālī Manuscript* solves it in two steps by the direct Rule of Three thus:

(i) If 1 person lives on 8 *drammas*, 70 persons live on how much?
The three terms are 1 / 8 / 70 and the result $70 \times 8 \div 1 = 560$ *drammas*.

(ii) If (70 persons can live) on 560 for 42 days, 8 *drammas* will last how many days?
The terms are 560 / 42 / 8 and the result $8 \times 42 \div 560 = 3/5$ days.

This is interesting because this is precisely the procedure adopted in the commentaries of other texts for verifying the results of extended problems with five terms such as barter, mixtures and the like.[32]

Rule of Three in the Islamic World

About the transmission of the Rule of Three to the Islamic world and thence to Europe, I have nothing to add, but I will summarise the little I know and raise one or two questions. We have seen that by the time of Brahmagupta in the early seventh century, the Rule of Three and its variations reached full development. In the next century, various elements of Indian mathematics and astronomy were disseminated to the Islamic world. The Rule of Three seems to be one of these elements thus transmitted. From the ninth century onwards, Arab mathematicians began to discuss the Rule of Three and other variants [Ansari & Hussain 1994: 223].

Thus al-Khwārizmī (ca. 850) discusses the Rule of Three in his book on Algebra. This treatise contains a small chapter on commerical problems including the

32 See, for example, [*Buddhivilāsinī*: 83 (barter), 91 (cistern problem), etc.].

simple Rule of Three according to the Indian model [Juschkewitsch 1964: 204]. We have noted al-Bīrūnī's (973–1048) reference to the Inverse Rule of Three. He also composed an exclusive tract on the Rule of Three entitled *Fī Rāshikāt al-Hind* [*al-Bīrūnī*]. Here he discusses direct and inverse Rule of Three as well as the rules for five, seven and more terms up to seventeen [Juschkewitsch 1964: 214].

But I do not quite know whether the Arab texts discuss all the Indian variations. Regarding the method of setting down the terms, al-Bīrūnī arranged them in two vertical columns, even for the Rule of Three. But this vertical arrangement did not reach Europe, where emphasis is laid on the horizontal arrangement of the three terms as in Indian *Trairāśika*. So there must be at least one Arabic source which retained the Indian system of arranging the three terms in a horizontal row and transmitted it to the West. Again, European sources prescribe a method of verification for the Rule of Three that is not found in Sanskrit texts. Do the Arabic texts have such a system?

Rule of Three in Europe

The story of the dissemination of Indian numerals as well as commercial problems to Europe, especially through the mediation of Leonardo of Pisa, is well documented as is the praise of the Rule of Three as the Golden Rule [Tropfke 1930: 190–191; Juschkewitsch 1964: 120; Smith 1925: 486 ff.; Wagner 1988: 181]. The Inverse Rule of Three was also known in Europe [Smith 1925: 490–491], but other variations such as the Rules of Five, Seven, etc., do not seem to have enjoyed as much popularity as the direct Rule of Three. Even when these variations did occur, they were not solved in the way they were solved in India. Recall the lion in the pit problem and its solution by Leonardo.

The only original source that I had access to, physically and linguistically, is the *Bamberger Rechenbuch* (1483) attributed to Ulrich Wagner. Three points struck me in this book as remarkable: (i) Stress is laid on the linear arrangement of the three terms;[33] (ii) If one of the terms is unity, it is expressely stated not to multiply with it or divide by it as the case may be;[34] (iii) Verification of the Rule of Three: Interchange the positions of the argument and requisition, set down the result in the middle, and apply the Rule of Three. The new result must be the same as the old fruit.[35] That is to say, set down C, D, A; $A \times D \div C = B$.

33 [*Bamberger Rechenbuch*: 181–182]: "Und sind es drei Dinge, die du setzt. Darunter muss das erste und letzte allemal gleich sein. Und zur letzten sollst du setzen was du wissen willst. Dasselbe und das mittlere sollst du miteinander multiplizieren und durch das erste teilen."

34 [*Bamberger Rechenbuch*: 183]: "...wenn 1 am letzten steht, so multipliziere nicht, ... Wenn aber 1 an der ersten Stelle steht, so teils nicht in das erste, ..."

35 [*Bamberger Rechenbuch*: 187]: "Willst du probieren, was du mit der Goldnen Regel gemacht hast, so kehre es um. Also, was du an der ersten Stelle gehabt hast, setze an die dritte. Und was an der dritten Stelle gestanden ist, setze an die erste. Und was kostet, an die Mitte. Und dann mache es nach der oben mitgeteilten Regel. Und es muss gerade soviel kommen, wie vorher in der Mitte gestanden ist."

Conclusion

We have seen that in the Rule of Three and other rules, it is the mechanical arrangement of the given quantities that is important, as is also the equally mechanical process of solution. Thus in India, the terms in the Rule of Three are generally written in a horizontal row, while those in the Rules of Five, etc., are set down in two vertical columns. It is possible to conceive of an intermediate stage when the terms of the Rule of Three were also written in two vertical columns as in the Rule of Five, etc.

It was probably Brahmagupta who introduced the custom of writing the terms in two vertical columns. One can argue that Brahmagupta employs both styles of writing. In [*Brāhmasphuṭasiddhānta* 12.10] the terms of the Rule of Three are to be set down in a horizontal sequence, so also the terms of the Inverse Rule of Three in [*Brāhmasphuṭasiddhānta* 12.11ab]. According to [*Brāhmasphuṭasiddhānta* 12.11cd–12], however, the terms are to be set down in two columns for all rules with odd terms, from three to eleven.

Thus, for the following sum there are many ways of setting down terms: If 5 mangoes cost 10 rupees, how much do 8 mangoes cost?

5	10	8

5
10
8

5	8
10	

5	8
10	0

In spite of the high encomiums it received in India and Europe, the Rule of Three, or more specifically the method of solution proposed in India for the Rule of Three, does not seem to be in use anywhere. In Europe, it was dropped from the school books in the last century. Possibly it was dropped in India also in the last century when modern mathematics was introduced here.

In the north Indian schools, I am told, the problems related to the Rule of Three are solved today by what is known as "One-One-Rule" (*ek-ek-niyam*). What it is will be clear from the following example:

5 mangoes cost	10 rupees
1 mango	10/5 rupees
8 mangoes	$10/5 \times 8 = 16$ rupees.

It is, of course, the more logical method. But when I went to school in south India, more than half a century ago, we skipped the middle line. We first wrote the proposition thus:

5 mangoes cost	←	10 rupees
8 mangoes	↗	?

The answer, we were taught, is obtained by multiplying the 8 by the second term in the first line and then dividing the product by the first term in the first line. One drew an arrow diagonally from 8 to 10 and another arrow horizontally from 10 to 5. It was thus a compromise between the one line system of old and the three line system of modern times. I still practice this method. I must, however, confess that until now I never asked where it came from.

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